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PRINCIPLES OF ERROR THEORY AND CARTOGRAPHIC APPLICATIONS

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PRINCIPLES OF ERROR THEORY AND CARTOGRAPHIC APPLICATIONS

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NOTICES

This report is issued to disseminate timely, useful technical data to activities engaged in cartography, geodesy, and related subjects. Nothing in this report is to be construed as necessarily coinciding with USAF doctrine.

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PREFACE

Optimum utilization of ACIC research and production requires that the accuracy of source material, interim and final products be considered. The accuracy is expressed by an error statement which indicates whether the product is reliable and acceptable or should be used with discretion. Therefore, the error statement must be representative of the product and have a sound statistical basis. The purpose of this paper is to present and explain the theory and procedures for providing a valid and meaningful error statement.

The normal distribution of linear errors is explained in detail because two and three-dimensional error distributions are more easily analyzed statistically by individual treatment of the linear components. The principles of the linear error distribution apply only to independent random errors, assuming that systematic errors have been eliminated or reduced sufficiently to permit treatment as random errors.

Although a truly circular or spherical error distribution seldom occurs in a sample of observations, the concepts are desirable for ease of computation and understanding. Consequently, considerable attention is given to the computation of an approximate circular or spherical error distribution from unequal linear components of a two or three-dimensional error distribution, yet retaining properties such as precision indexes of the truly circular and spherical distributions. Some characteristics of circular and spherical error distributions differ from those of the linear error distribution; however, the distinction is of an academic nature and hence is not emphasized in the text.

Organizations using ACIC charting products should find the discussion helpful in interpreting statements of cartographic accuracy. The formulas and principles can also be applied to weapon system accuracy evaluation and other purposes provided that the assumption of a normal distribution of independent random variables is feasible.

Important functions and equations are presented in the text, while lengthy derivations are relegated to appendixes. Liberal numbers of references are inserted after major headings to facilitate further study.

ABSTRACT

One of the most useful contributions of error theory is the precision index which identifies the error distribution and specifies the probability that the true error in a quantity does not exceed a certain value. This situation is applicable to the evaluation of map and geodetic information, in that it makes possible meaningful accuracy statements having uniform interpretation, and is compatible with established map accuracy standards which specify limits of permissible error in various categories. Standardized procedures and supporting theory for computing linear, circular, and spherical precision indexes are presented. The recommended procedure for computing the circular or spherical standard error from linear standard errors in X and Y, or X, Y, and Z directions, respectively, is to average the linear standard errors. Other precision indexes in the same error distribution are easily computed from the linear, circular, and spherical standard errors -- the most important precision indexes.

1. ONE-DIMENSIONAL (LINEAR) ERRORS

1.1. Introduction. Various aspects of the sciences of geodesy, cartography, and photogrammetry involve the measurement of physical quantities and the utilization of such measurements. Regardless of the precision of the instrument, no measurement device or method gives the true value for the quantity measured. Mechanical imperfections in instruments and the limitations introduced by human factors are such that repeated measurements of the same quantity result in different values. Variations among successive values are caused by errors[1] in the observations.

While the theory of errors does not yield a true value nor improve the quality of observations, it does provide a way of estimating the most probable value for the quantity and of determining the certainty attributable to the estimate. Once this has been established, a least squares adjustment can be used to remove or distribute the observational errors to obtain a solution which is relatively free of discrepancies.

1.2. Classes of Errors. (ref. 6, 19, 22) Errors fall into three general classes which may be categorized by origin as (1) blunders, (2) systematic, and (3) random.

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[1] The true error of each observation is the difference between the true value of a quantity and the measured value.

Blunders are mistakes caused by misreading scales, transposing figures, erroneous computations, or careless observers. They are usually large and easily detected by repeated measurements. Systematic errors follow some fixed law and are generally constant in magnitude and/or sign within a series of observations. The origin of systematic errors in geodetic measurements is primarily within the instrument or measuring device. Causes of systematic error include faulty instrument calibration, errors inherent in the graduation of scales, and changes in performance resulting from variations in temperature and humidity. Systematic errors can be eliminated or substantially reduced when the cause is known. Random errors are those remaining after blunders and systematic errors have been removed. They result from accidental and unknown combinations of causes beyond the control of the observer. Random errors are characterized by: (1) variation in sign — positive and negative errors occurring with equal frequency, (2) small errors occurring more frequently than large errors, and (3) extremely large errors rarely occurring.

The probability that a random error will not exceed a certain magnitude may be inferred from an analysis of the normal or Gaussian distribution of an infinite number of random variables.

1.3. Basic Concepts of Probability. (ref. 2, 3) Probability is defined as the frequency of occurrence in proportion to the number of possible occurrences, or simply, the ratio of the number of successes to the number of trials. Let A and B symbolize two completely independent events. Denote P(A) as the probability of the event "A" and P(B) as the probability of the event "B". The probability of any event happening must be between zero and one.[1] That is, zero probability means that the particular event will never take place, and a probability of one means that the particular event will occur each trial. For example, the probability of rolling the number 7 with a single die is 0.0 (an impossible event), but the probability of rolling a number from and including 1 through 6 is 1.0.

Rule 1. The probability of event A is equal to or greater than 0 but equal to or less than 1.

$$0 \leq P(A) \leq 1$$

Rule 2. The probability of a failure, or the probability of an event not occurring, is 1 minus the probability that it will occur.

$$1 - P(A) = \text{failure of event A}$$

Rule 3. The probability of either of two events A or B occurring is equal to the sum of their individual probabilities.

$$P(A \text{ or } B) = P(A) + P(B)$$

---

[1] Probability is also denoted by a percentage.

An example is the probability of either a 3 or 4 occurring on the single roll of a die:

$$P(3 \text{ or } 4) = 1/6 + 1/6 = 1/3$$

Rule 4. The probability of two events occurring simultaneously is equal to the product of their individual probabilities.

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

An example is the probability of both A = 3 and B = 4 occurring in a single roll of 2 dice:

$$P(3 \text{ and } 4) = 1/6 \cdot 1/6 = 1/36$$

The probabilities of occurrence of the numbers summed from each of 36 possible combinations resulting from the single roll of two dice are presented in Figure 1. The probability of rolling the number 7, for example, is 6/36 or 1/6 since there are six combinations which have a sum of seven. A histogram[1] of the data approximates the area under a superimposed smooth curve. If the number of dice in a single roll were increased, the histogram would rapidly approach the smooth curve, called the normal probability density curve.

[1] A column is constructed for each number by blocks each representing an area equal to 1/36 probability.

Number	Probability[1]	Combinations
1	0	
2	1/36	(1,1)
3	2/36	(1,2) (2,1)
4	3/36	(1,3) (3,1) (2,2)
5	4/36	(1,4) (4,1) (2,3) (3,2)
6	5/36	(1,5) (5,1) (2,4) (4,2) (3,3)
7	6/36	(1,6) (6,1) (2,5) (5,2) (4,3) (3,4)
8	5/36	(2,6) (6,2) (3,5) (5,3) (4,4)
9	4/36	(3,6) (6,3) (4,5) (5,4)
10	3/36	(4,6) (6,4) (5,5)
11	2/36	(5,6) (6,5)
12	1/36	(6,6)
13	0	

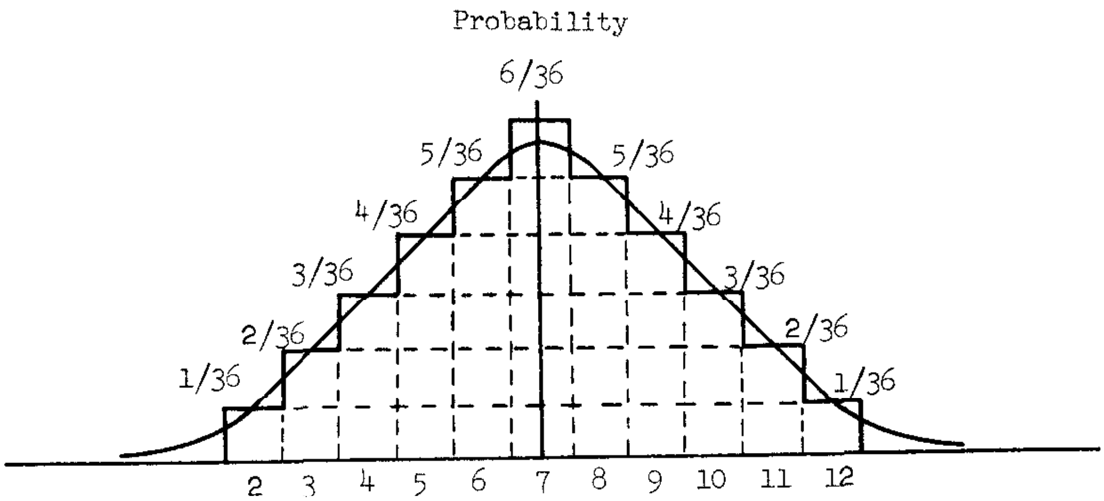


Figure 1

Probabilities of Numbers from the Roll of Two Dice

[1] Note that the sum of the probabilities is 1.

1.4. The Normal Distribution of a Continuous Random Variable. (ref. 3, 24) The area under the normal probability density curve (Figure 2a) represents the total probability of the occurrence of the continuous random variable $\underline{x}$ and is equal to one, or 100%. The mathematical expression of the curve is the normal probability density function, $p(\underline{x})$:

$$p(\underline{x}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\underline{x}-\mu)^2}{2\sigma^2}} \qquad (1\text{-}1)$$

where:

$\underline{x}$ = the random variable

μ = a parameter representing the mean value of $\underline{x}$

σ = a parameter representing the standard deviation, a measure of the dispersion of the random variable from the mean, μ. (The square of the standard deviation is called the variance.)

$\sqrt{2\pi}$ = 2.5066 . . .

e = the base of natural logarithms, 2.71828...

The parameters are computed from an infinite number of random variables:

$$\mu = \frac{\sum_{i=1}^{n} \underline{x}_i}{n} \qquad (1\text{-}2)$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (\underline{x}_i - \mu)^2}{n}} \tag{1-3}$$

where:

n = the number of random variables, and

$n \rightarrow \infty$.

The normal probability <u>distribution</u> function determines the probability that the random variable will assume a value within a certain interval and is derived from the normal probability <u>density</u> function by integrating between limits of the desired interval. Letting the limits range from $-\infty$ to $\underline{x}$:

$$P(\underline{x}) = \int_{-\infty}^{\underline{x}} p(\underline{x}) \, d\underline{x}$$

$$P(\underline{x}) = \int_{-\infty}^{\underline{x}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\underline{x} - \mu)^2}{2\sigma^2}} \, d\underline{x} \tag{1-4}$$

The value of $P(\underline{x})$ ranges between 0 and 1, illustrated in Figure 2b. As $\underline{x}$ approaches its upper limit, $P(\underline{x})$ approaches 1; as $\underline{x}$ approaches its lower limit, $P(\underline{x})$ approaches zero. This is true since $\underline{x}$ cannot exceed nor be less than its defined limits.

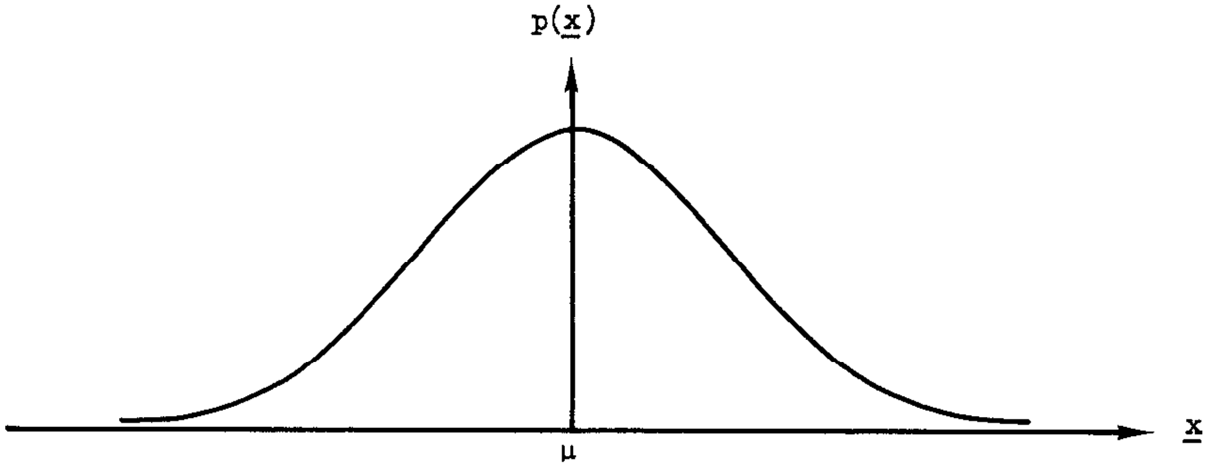


Figure 2a

Normal Probability Density Curve

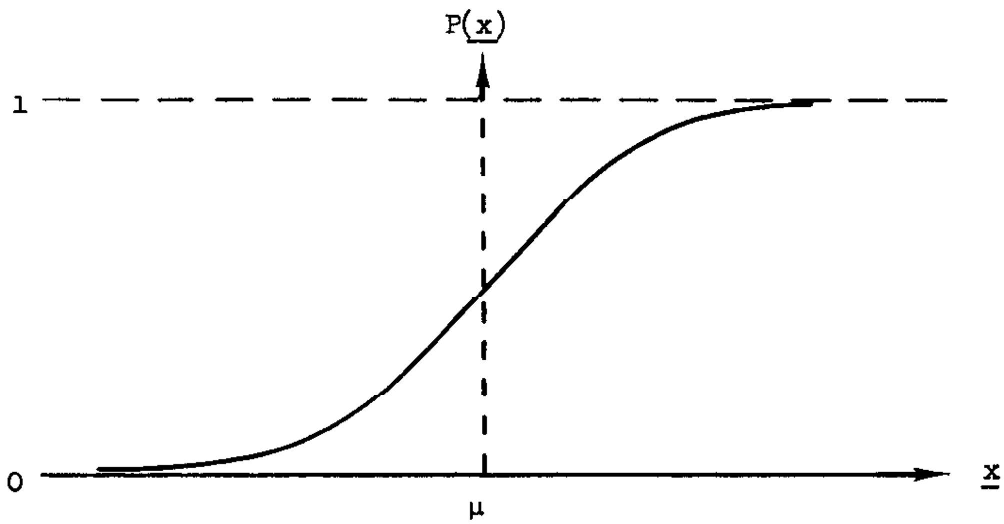


Figure 2 b

Normal Probability Distribution Curve

1.5. Application of the Probability Density Function to Random Errors. (ref. 3, 15, 21, 22) The normal probability density curve of an infinite number[1] of measurements of the unknown quantity X is expressed by parameters analogous to those of equation (1-1). The true value μ_X is the mean of the distribution of the observed values $X_1, X_2, X_3 \cdots X_n$. The curve, illustrated in Figure 3, has the mathematical form:

$$p(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(X_i - \mu_X)^2}{2\sigma^2}} \tag{1-5}$$

where: $\sigma = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \mu_X)^2}{n}}$

The normal probability density curve of errors has a mean of zero and is identical in form to that of the observed values. Illustrated in Figure 4, the curve is described by the function:

$$p(\epsilon) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2\sigma^2}} \tag{1-6}$$

where: ϵ = the true error;
$\epsilon = X_i - \mu_X$

σ = the standard deviation of the errors, hereafter designated the standard error;

$$\sigma^2 = \frac{\sum_{i=1}^{n} \epsilon_i^2}{n}$$

---

[1] The population or universe in statistics.

Since the true value of a quantity cannot be measured and an infinite number of measurements is impractical, estimated values obtained from a finite number or sample[1] of measurements must be substituted for the true value and the parameters of the density function. The most probable value ($\overline{X}$) approximates the true value and is determined from the arithmetic mean[2] of observed values:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} \tag{1-7}$$

The true error is approximated by the residual "x"[3], hereafter designated the error and defined as the difference between the observed value and the most probable value:

$$x = X_i - \overline{X} \tag{1-8}$$

The standard error computed from a sample (σ_x) is identified by a subscript[4] and computed from:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}} = \sqrt{\frac{\sum x^2}{n-1}} \tag{1-9}$$

---

[1] As the number of measurements in the sample becomes larger, the reliability of the estimate increases. Often, 30 values provide an adequate estimate.

[2] See Appendix B.

[3] The residual is represented by "v" in some texts.

[4] The standard error derived from a sample is designated in some texts by "s" or "m".

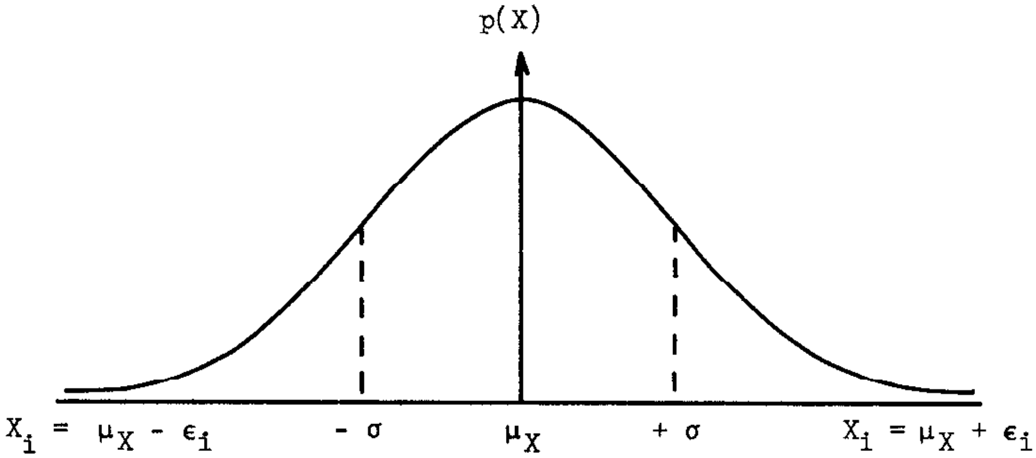
The normal probability density function of errors now becomes:

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}} \qquad (1\text{-}10)$$

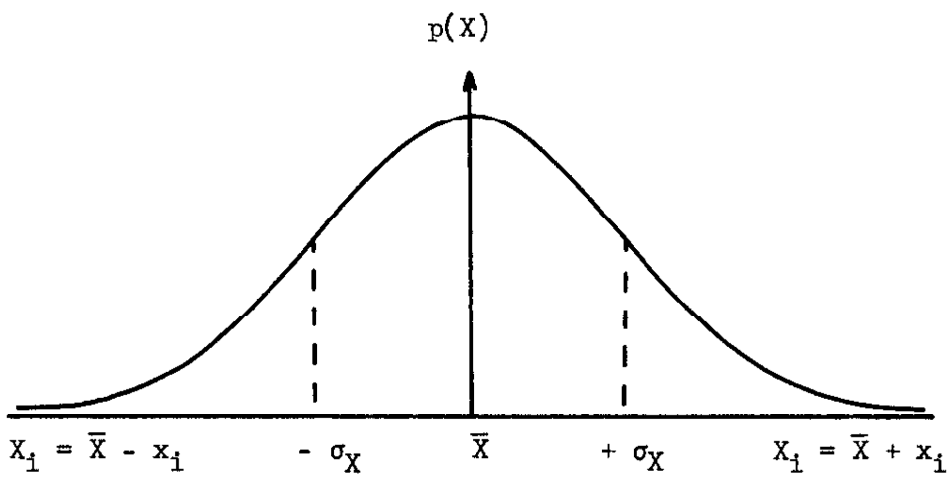
The parameters $\overline{X}$ and σ_x may assume different values as various samples are selected from the same population and are, therefore, random variables with dispersion expressed by similar parameters. The standard error of the mean, $\sigma_{\overline{X}}$, and the standard error of the standard error, σ_σ , indicate the reliability of the estimate and help "round off" the computed values:

$$\sigma_{\overline{X}} = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n\,(n-1)}} = \frac{\sigma_x}{\sqrt{n}} \qquad (1\text{-}11)$$

$$\sigma_\sigma = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{2\,(n-1)^2}} = \frac{\sigma_x}{\sqrt{2\,(n-1)}} \qquad (1\text{-}12)$$



Population



Sample

Figure 3

Normal Probability Density Curve of Observed Values

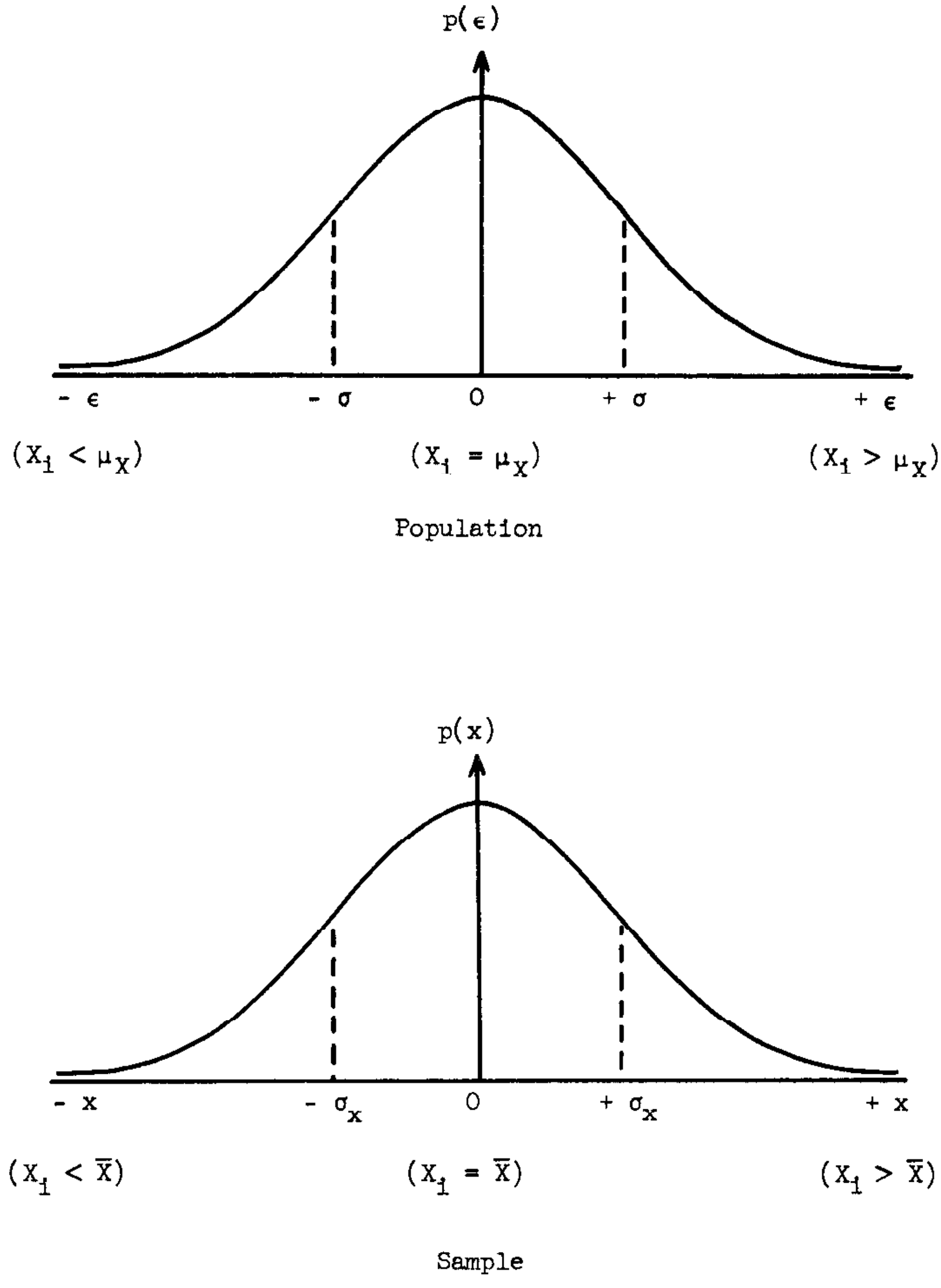


Figure 4

Normal Probability Density Curve of Errors

1.6. Precision Indexes. (ref. 3, 22) A precision index reveals how errors are dispersed or scattered about zero and reflects the limiting magnitude of error for various probabilities. For example, 50% of all errors in a series of measurements do not exceed ±20 feet; 90% do not exceed ±49 feet. Although different errors are given, each expresses the same precision of the measuring process (Figure 5). The standard error and average error (η) are two indexes with theoretical derivations. Common usage has included three additional probability levels which are, in effect, precision indexes: (1) probable error (PE), (2) map accuracy standard (MAS), and (3) the three sigma error (3σ).

The standard error is the most important of the indexes and has the probability of:

$$P(x) = \int_{-\sigma_x}^{+\sigma_x} p(x) \; dx = 0.6827 \qquad (1\text{-}13)$$

Or, 68.27% of all errors will occur within the limits of $\pm \sigma_x$.

The average error is defined as the mean of the sum of the absolute values of all errors:

$$\eta = \frac{\sum_{i=1}^{n} |(x_i - \bar{x})|}{n} = \frac{\sum |x|}{n} \qquad (1\text{-}14)$$

The probability represented by the average error is 0.5751, or 57.51%. The average error is easily computed from the standard error:

$$\eta = 0.7979 \; \sigma_x \qquad (1\text{-}15)$$

The probable error is that error which 50% of all errors in a linear distribution will not exceed. Specifically, the true error is equally likely to be larger or smaller than the probable error. Expressed mathematically:

$$PE = \int_a^b p(x) \, dx = 0.50 \qquad (1\text{-}16)$$

The probable error is computed from the standard error:

$$PE = 0.6745 \sqrt{\frac{\sum x^2}{n - 1}} = 0.6745 \, \sigma_x \qquad (1\text{-}17)$$

The U.S. National Map Accuracy Standards specify that no more than 10% of map elevations (a one-dimensional error) shall be in error by more than a given limit. The standards are commonly interpreted as limiting the size of error of which 90% of the elevations will not exceed. Therefore, the map accuracy standard is represented by:

$$MAS = \int_a^b p(x) \, dx = 0.90 \qquad (1\text{-}18)$$

or, computed from the standard error:

$$MAS = 1.6449 \, \sigma_x \qquad (1\text{-}19)$$

The three sigma error, as the name implies, is an error three times the magnitude of the standard error. The 3σ error is used because it approaches near-certainty — 0.9973 or 99.73% probability.

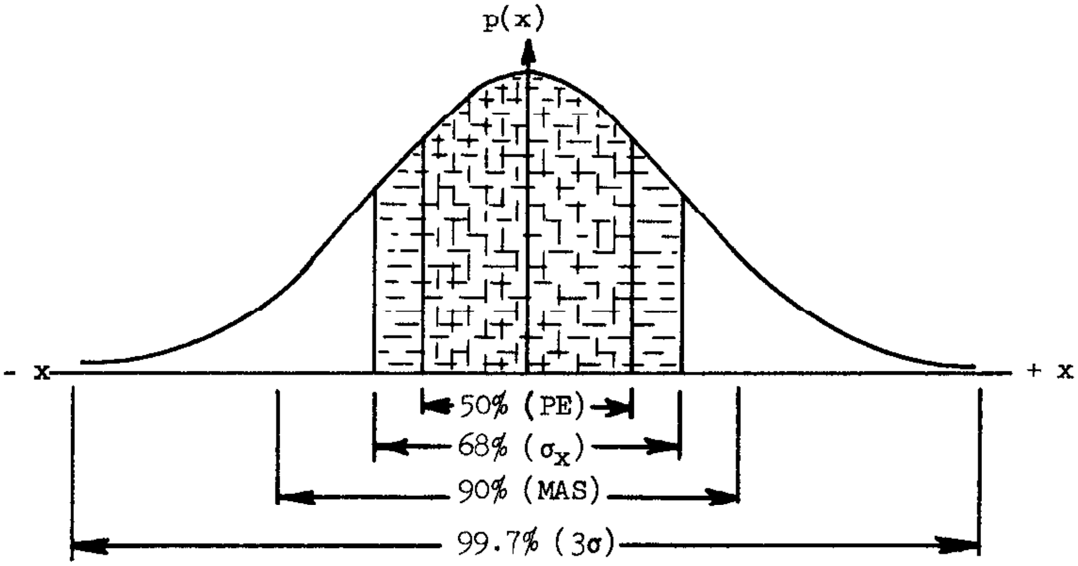


Figure 5

Probability Areas

1.7. <u>Conversion Factors.</u> (ref. 20, 27) Since all precision indexes are related to the standard error (Table I), factors computed from this relationship (Table II) will convert the error at a given probability to the error at another probability.

Table I

Summary of Linear Precision Indexes

Symbol	Probability	Derivation
PE	.5000	0.6745 σ_x
η	.5751	0.7979 σ_x
σ_x	.6827	1.0000 σ_x
MAS	.9000	1.6449 σ_x
3σ	.9973	3.0000 σ_x

Table II

Linear Error Conversion Factors

From \ To	50.00%	57.51%	68.27%	90.00%	99.73%
50.00%	1.0000	1.1830	1.4826	2.4387	4.4475
57.51%	0.8453	1.0000	1.2533	2.0615	3.7599
68.27%	0.6745	0.7979	1.0000	1.6449	3.0000
90.00%	0.4101	0.4851	0.6080	1.0000	1.8239
99.73%	0.2248	0.2660	0.3333	0.5483	1.0000

1.8. Propagation of Errors. (ref. 5, 29) A quantity f_1 is computed from two measured quantities a and b, where f(a,b) denotes a function of a and b. The error Δf of f_1 is affected by the errors in both a and b: Δa and Δb. Assuming a and b are independent, and the errors Δa, Δb are randomly distributed, the combined error Δf can be computed by the general equation:

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial f}{\partial b}\right)^2 \sigma_b^2} \qquad (1\text{-}20)$$

where: σ_f = the standard error of f

σ_a, σ_b = the standard errors of a and b

$\frac{\partial f}{\partial a}$, $\frac{\partial f}{\partial b}$ = partial derivatives of f, with respect to a and b.

Application of the general equation to specific conditions produces the following rules[1] for the function f(a,b):

Rule 1. Sum and Difference

$$f = (a + b) \quad \text{or } f = (a - b)$$

$$\sigma_f = \sqrt{\sigma_a^2 + \sigma_b^2} \qquad (1\text{-}21)$$

[1] Derivations in Appendix C.

Rule 2. Product of Factors

$$f = a^m b^q$$

$$\frac{\sigma_f}{f} = \sqrt{m^2 \left(\frac{\sigma_a}{a}\right)^2 + q^2 \left(\frac{\sigma_b}{b}\right)^2} \qquad (1\text{-}22)$$

Rule 3. Simple Product or Quotient

$$f = a \cdot b \quad \text{or} \quad f = a/b$$

$$\frac{\sigma_f}{f} = \sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2} \qquad (1\text{-}23)$$

Indexes other than the standard error can be used to propagate errors. For example, using Rule 1:

$$(PE)_f = \sqrt{(PE)_a^{\,2} + (PE)_b^{\,2}}$$

$$\eta_f = \sqrt{\eta_a^{\,2} + \eta_b^{\,2}}$$

$$\sigma_f = \sqrt{\sigma_a^{\,2} + \sigma_b^{\,2}}$$

$$(MAS)_f = \sqrt{(MAS)_a^{\,2} + (MAS)_b^{\,2}}$$

and $$(3\sigma)_f = \sqrt{(3\sigma)_a^{\,2} + (3\sigma)_b^{\,2}}$$

However, note that the index must be consistent throughout the formula. That is:

$$(PE)_f \neq \sqrt{(PE)_a^{\,2} + \sigma_b^{\,2}}$$

$$\eta_f \neq \sqrt{(MAS)_a^2 + (3\sigma)_b^2}$$

$$\sigma_f \neq \sqrt{\sigma_a^2 + \eta_b^2}$$

etc.

1.9. Examples of Linear Errors. The foregoing discussion demonstrates the use of the normal distribution in the analysis of random errors. There are numerous opportunities for the occurrence of random variables in cartographic and geodetic work. For example, the base lines and measured angles, observed lengths of lines, elevations, etc., resulting from geodetic triangulation, traverse, and leveling all contain error. The same is true of celestial and gravimetric observations as well as distances measured by trilateration. The principles of error theory can be used advantageously to analyze the results in terms of the specifications established for the survey.

In ACIC, the normal linear error distribution has important applications with respect to evaluating the accuracy of positional information. In addition to the one-dimensional errors which exist in such positional data as elevations above mean sea level, the linear error components of two and three-dimensional positions can be analyzed by applying principles of the normal linear error distribution. The following sections contain discussions of the utility of the linear standard error for analyzing two and three-dimensional distributions.

2. TWO-DIMENSIONAL (ELLIPTICAL, CIRCULAR) ERRORS

2.1. <u>Introduction</u>. A two-dimensional error is the error in a quantity defined by two random variables. For example, consider the true geographic position of a point referred to X and Y axes. Each observation of the X and Y coordinates will contain the errors "x" and "y". When assumed <u>random</u> and <u>independent</u>, each error has a probability density distribution of:

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}}$$

and:

$$p(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_y^2}}$$

Applying Rule 4 of Section 1.3., the two-dimensional probability <u>density</u> function becomes:

$$p(x,y) = \frac{1}{2\pi\ \sigma_x\ \sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)} \qquad (2\text{-}1)$$

Rearranging terms:

$$p(x,y)\ \sigma_x\sigma_y\ 2\pi = e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

Therefore:

$$- 2 \ln \left[p(x,y) \quad \sigma_x \, \sigma_y \, 2\pi \right] = \frac{x^2}{\sigma_x^{\,2}} + \frac{y^2}{\sigma_y^{\,2}} \tag{2-2}$$

For given values of p(x,y), the left side of equation (2-2) is a constant K^2

Then:

$$K^2 = \frac{x^2}{\sigma_x^{\,2}} + \frac{y^2}{\sigma_y^{\,2}} \tag{2-3}$$

For values of p(x,y) from 0 to ∞, a family of equal probability density ellipses are formed with axes $K \sigma_x$ and $K \sigma_y$.

When $\sigma_x = \sigma_y$, equation (2-2) becomes:

$$-2\sigma_x^{\,2} \ \ln \left[p(x,y) \ \sigma_x^{\,2} \ 2\pi \right] = x^2 + y^2 \tag{2-4}$$

For a given value of p(x,y), the left side of equation (2-4) is a constant which is the square of the radius of an equal probability density circle.

The probability density function integrated over a certain region becomes the probability <u>distribution</u> function which yields the probability that x and y will occur <u>simultaneously</u> within that region, or:

$$P(x,y) = \iint p(x,y) \, dx \, dy$$

However, since both positive and negative values of either "x" or "y" will occur with equal frequency, the errors may be considered as radial errors, designated by "r", where $r = \sqrt{x^2 + y^2}$.

2.2. Elliptical Errors. (ref. 15, 20) The probability of an ellipse is given by the distribution function:

$$P(x,y) = 1 - e^{-\frac{K^2}{2}} \qquad (2\text{-}5)$$

The solution of equation (2-5) with values of K for different probabilities yields the results shown in Table III. For a 39% probability, the axes of the ellipse are 1.0000 σ_x and 1.0000 σ_y; for a 50% probability, the axes are 1.1774 σ_x and 1.1774 σ_y.

Table III

Values of the Constant K

Probability	K
39.35%	1.0000
50.00%	1.1774
63.21%	1.4142
90.00%	2.1460
99.00%	3.0349
99.78%	3.5000

The use of the error ellipse is complicated by the problem of axes orientation and propagation of elliptical errors. Therefore, the ellipse is commonly replaced by a circular form which is easier to use and understand.

2.3. Circular Errors.

2.3.1. Circular Probability Distribution Function. (ref. 1, 24) The probability distribution function of the radial error expressing the probability that "r" will be equal to or less than radius R, or the probability that the vector xy will be contained within a circle of radius R, is derived in Appendix D and stated as:

$$P(R) = \frac{1}{\sigma_x \sigma_y} \int_0^R r e^{-\frac{r^2}{4\sigma_y^2}\left[1 + \frac{\sigma_y^2}{\sigma_x^2}\right]} I_o\left[\frac{r^2}{4\sigma_y^2}\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right] dr \qquad (2\text{-}6)$$

A special case of the P(R) function (2-6) is formed when r=R, and $\sigma_x = \sigma_y = \sigma_r \equiv \sigma_c$. From Appendix D, part 2:

$$P(R) = P_c = 1 - e^{-\frac{R^2}{2\sigma_c^2}} \qquad (2\text{-}7)$$

where:

P_c = the circular probability distribution function, a special case of P(R)

R = the radius of the probability circle

σ_c = the circular standard error, a special case of σ_r when $\sigma_r = \sigma_x = \sigma_y$.

When σ_x and σ_y are not equal, the P(R) function, (2-6), is modified by letting "a" equal the ratio $\frac{\sigma_x}{\sigma_y}$ where σ_x is the smaller standard error of the two. Then from Appendix D, part 3:

$$P(R) = \frac{2a}{1 + a^2} \int_0^x e^{-v} \; I_o \, (vk) \; dv \qquad (2\text{-}8)$$

where:

$$x = \frac{R^2}{4\sigma_y^2} \left[\frac{1 + a^2}{a^2}\right]$$

$$v = \frac{r^2}{4\sigma_y^2}\left[\frac{1 + a^2}{a^2}\right]$$

$$k = \left(\frac{1 - a^2}{1 + a^2}\right)$$

Equation (2-8) can be solved[1] for different probabilities or values of P(R) representing precision indexes of the error distribution.

2.3.2. Circular Precision Indexes. (ref. 19, 20, 27) The precision indexes illustrated in Figure 6 are measures of the dispersion of errors in a distribution and represent the error which is unlikely to be exceeded for a given probability. The preferred circular precision indexes, consistent with indexes used in the linear distribution, are: (1) the circular standard error (σ_c), (2) the circular probable error (CPE, CEP), (3) the circular map accuracy standard (CMAS), and (4) the circular near-certainty error, three-five sigma (3.5 σ_c). The mean square positional error (MSPE), an additional index which has been used at ACIC, is not recommended because the probability represented varies when σ_x and σ_y are not equal.

The probability of the circular standard error is found by solving equation (2-7) for P_c when $\sigma_c = R$, thus:

$$P_c = 1 - e^{-\frac{\sigma_c^2}{2\sigma_c^2}}$$

[1] Described in Appendix D, part 4.

$$= 1 - e^{-\frac{1}{2}}$$

$$= 1 - 0.60653$$

$$\therefore P_c = 0.3935 \qquad (2\text{-}9)$$

That is, 39.35% of all errors in a circular distribution are not expected to exceed the circular standard error.

For a truly circular distribution, the linear standard errors are equal and identical to the circular standard error ($\sigma_x = \sigma_y \equiv \sigma_c$). When σ_x and σ_y are not equal, a normal circular error distribution may be substituted for the elliptical distribution. The substitution is satisfactory for error analysis within specified $\sigma_{min}/\sigma_{max}$[1] ratios. Because of distortion in the error distribution[2] for low ratios, however, the circular concept should be used with discretion.

An approximate circular standard error is determined from equation (2-8) by letting $P(R) = 39.35\%$ and $R = \sigma_c$. Values of σ_c/σ_{max} for ratios of $\sigma_{min}/\sigma_{max}$ from 0.0 to 1.0 are contained in Table IV and plotted in Figure 7. For the $\sigma_{min}/\sigma_{max}$ ratio between 1.0 and 0.6, the curve is a straight line with the equation:

---

[1] Where σ_{min} is the minimum or smaller linear standard error of the two.

[2] See Appendix F.

$$\sigma_c \sim (0.5222\ \sigma_{min} + 0.4778\ \sigma_{max}) \tag{2-10}$$

A rapid approximation gives a slightly larger σ_c value for the same $\sigma_{min}/\sigma_{max}$ ratio:

$$\sigma_c \sim 0.5000\ (\sigma_x + \sigma_y) \tag{2-11}$$

As $\sigma_{min}/\sigma_{max}$ approaches zero, the 39.35% probability curve follows a transition from circular, through elliptical, to the linear distribution form.[1] The curve does not effectively represent a circular standard error for $\sigma_{min}/\sigma_{max}$ ratios less than 0.6 because it is not compatible with other circular precision indexes. For example, the factor 1.1774 converts a circular error at 39% probability to a circular error at 50% probability when $\sigma_{min}/\sigma_{max} = 1.0$, but when $\sigma_{min} = 0$, the factor converting a linear error at 39% probability to a linear error at 50% probability is 1.3094.[2] The circular standard error computed from equation (2-11), however, can be converted to other circular precision indexes by constant circular conversion factors[3] for $\sigma_{min}/\sigma_{max}$ ratios between 1.0 and 0.2 and is, therefore, the preferred method for approximating the circular standard error.

---

[1] When $\sigma_{min} = 0$, the factor 0.5151 converts a linear error at 68% probability to an error at 39.35% probability.

[2] The transition curves of conversion factors are shown in Figures 10 and 11.

[3] Presented in Section 2.3.3.

Although it is not recommended because of limited applicability and extra computation required, an approximate σ_c may be computed by an alternate method:

$$\sigma_c \sim \sqrt{\frac{\sigma_x^2 + \sigma_y^2}{2}} \qquad (2\text{-}12)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.8

The circular probable error is the circular error which 50% of all errors in a circular distribution will not exceed, or the value of R in equation (2-7) which makes $P_c = 0.5$. The CPE (or CEP) in a truly circular distribution (i.e. $\sigma_x = \sigma_y = \sigma_c$) is computed by:

$$0.5 = 1 - e^{-\frac{R^2}{2\sigma_c^2}}$$

$$1 - 0.5 = e^{-\frac{R^2}{2\sigma_c^2}}$$

$$\ln 0.5 = -\frac{R^2}{2\sigma_c^2}$$

$$R^2 = 0.69315\ (2\sigma_c^2)$$

$$R = 1.1774\ \sigma_c$$

$$CPE = 1.1774\ \sigma_c \qquad (2\text{-}13)$$

When σ_x and σ_y are not equal, an approximate CPE is determined from equation (2-8) by letting P(R) = 50.00% and R = CPE. Values of CPE/σ_{max} for ratios of $\sigma_{min}/\sigma_{max}$ from 1.0 to 0.0 are tabulated in Table V. The 50% probability curve plotted in Figure 8 is approximated by a series of straight lines for different ratios of $\sigma_{min}/\sigma_{max}$ with the equations:

$$CPE \sim (0.6142\ \sigma_{min} + 0.5632\ \sigma_{max}) \tag{2-14}$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.3

$$CPE \sim (0.4263\ \sigma_{min} + 0.6196\ \sigma_{max}) \tag{2-15}$$

when $\sigma_{min}/\sigma_{max}$ is between 0.3 and 0.2

A rapid approximation of the CPE plots as a straight line which intersects the 50% probability curve at the point where $\sigma_{min}/\sigma_{max}$ = 0.2 and has the equation:

$$CPE \sim 0.5887\ (\sigma_x + \sigma_y) \tag{2-16}$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.2

The CPE computed by equation (2-16) is compatible with the circular standard error computed by equation (2-11)[1] and is, therefore, the preferred method for approximating the circular probable error within the specified limits.

---

[1] That is, the conversion factor of 1.1774 for converting σ_c to CPE is constant for ratios of $\sigma_{min}/\sigma_{max}$ between 1.0 and 0.2. Note that 1.1774 [0.5000 $(\sigma_x + \sigma_y)$] = 0.5887 $(\sigma_x + \sigma_y)$.

Although a circular error concept is not recommended for $\sigma_{min}/\sigma_{max}$ ratios less than 0.2, a near-linear 50% probability error may be computed to represent a CPE for lower ratios when a comparison of circular errors derived from different sources is required:

$$CPE \sim (0.2141\ \sigma_{min} + 0.6621\ \sigma_{max}) \quad \text{when } \sigma_{min}/\sigma_{max} \text{ is between 0.2 and 0.1} \tag{2-17}$$

$$CPE \sim (0.0900\ \sigma_{min} + 0.6745\ \sigma_{max}) \quad \text{when } \sigma_{min}/\sigma_{max} \text{ is between 0.1 and 0.0} \tag{2-18}$$

$$CPE \sim 0.6745\ \sigma_{max} \quad \text{when } \sigma_{min} = 0 \tag{2-19}$$

The following alternate methods of computing an approximate CPE are not recommended because of limited applicability:

$$CPE \sim 1.1774 \sqrt{\frac{\sigma_x^2 + \sigma_y^2}{2}} \tag{2-20}$$

$$\text{and } CPE \sim 0.8325 \sqrt{\sigma_x^2 + \sigma_y^2} \quad \text{when } \sigma_{min}/\sigma_{max} \text{ is between 1.0 and 0.8} \tag{2-21}$$

The <u>mean square positional error</u> (ref. 1, 11) is defined as the radius of the error circle equal to 1.4142 σ_c and has little significance in a truly circular error distribution. However, when σ_x and σ_y are approximately equal, the MSPE defines the error in a geographic position and is computed:

$$\text{MSPE} = \sqrt{\sigma_x^2 + \sigma_y^2} \quad \text{when } \sigma_{min}/\sigma_{max} \text{ is between 1.0 and 0.8} \tag{2-22}$$

The probability represented by the MSPE can be found by solving equation (2-7) for P_c, when R = MSPE and σ_c is approximated by equation (2-11), thus:

$$P_c = 1 - e^{-\frac{R^2}{2\sigma_c^2}}$$

$$P_c = 1 - e^{-\frac{(\sigma_x^2 + \sigma_y^2)}{2\sigma_c^2}} \tag{2-23}$$

When $\sigma_x = \sigma_y$:

$$P_c = 1 - e^{-1.0}$$

$$= 1 - 0.3679$$

$$P_c = 63.21\% \tag{2-24}$$

When $\sigma_x \neq \sigma_y$, the solution of (2-23) yields values of P_c (plotted in Figure 9) ranging from 64% when $\sigma_{min}/\sigma_{max}$ = 0.8 to 77% when $\sigma_{min}/\sigma_{max}$ = 0.3. Because of the variation in probability, the MSPE is not recommended for use as a precision index.

The circular map accuracy standard is based on the percentage level in use by the U.S. National Map Accuracy Standards which specify that no more than 10% of the well-defined points in a map will exceed a given error. The standards are commonly interpreted as limiting the size of error which 90% of the well-defined points will not exceed. Therefore, the circular map accuracy standard is represented by the value of R in equation (2-7) when $P_c = 0.90$, and is computed:

$$\text{CMAS} = 2.1460\ \sigma_c \tag{2-25}$$

or

$$\text{CMAS} = 1.8227\ \text{CPE} \tag{2-26}$$

The <u>three-five sigma error</u>, representing a circular probability of 99.78%, approaches near-certainty in a circular distribution and has a magnitude 3.5 times that of the circular standard error.

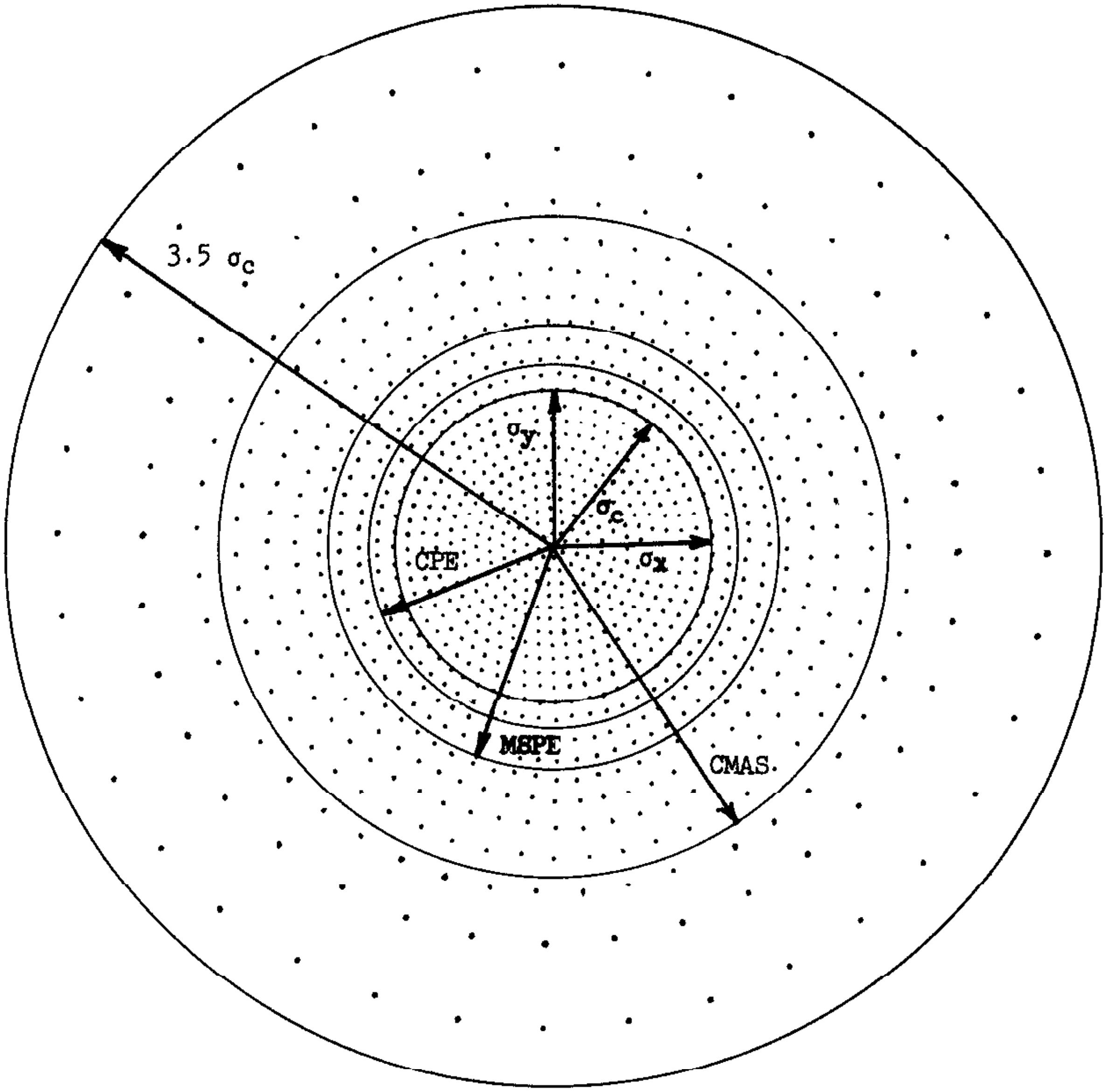


Figure 6

Normal Circular Distribution

Table IV

Solution of P(R) Function for P(R) = 39.35%

$\frac{\sigma_{min}}{\sigma_{max}}$	$\frac{\sigma_c}{\sigma_{max}}$
1.0000	1.0000
0.8165	0.9063
0.6547	0.8197
0.5000	0.7323
0.3333	0.6327
0.2294	0.5727
0.1005	0.5274
0.0	0.5151

Note: When P(R) = 39.35%, $R \sim \sigma_c$.

Figure 7

Curve of the P(R) Function When P(R) = 39.35%

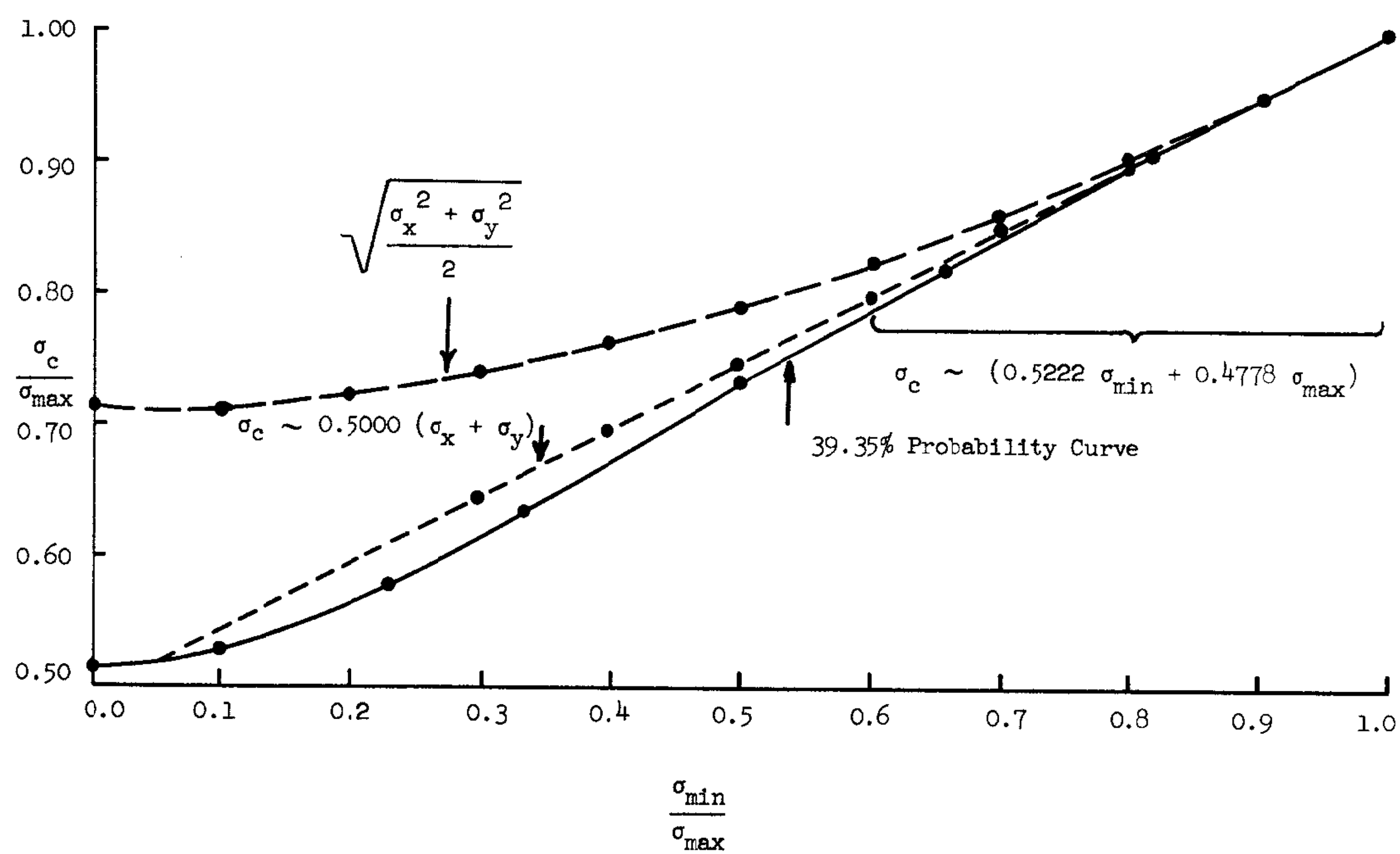


Table V

Solution of P(R) Function for P(R) = 50.00%

$\frac{\sigma_{min}}{\sigma_{max}}$	$\frac{CPE}{\sigma_{max}}$
1.000	1.1774
0.8165	1.0683
0.6547	0.9690
0.5000	0.8707
0.3333	0.7696
0.2294	0.7174
0.1005	0.6835
0.0	0.6745

Note: When P(R) = 50.00%, R ~ CPE

Figure 8

Curve of the P(R) Function When P(R) = 50.00%

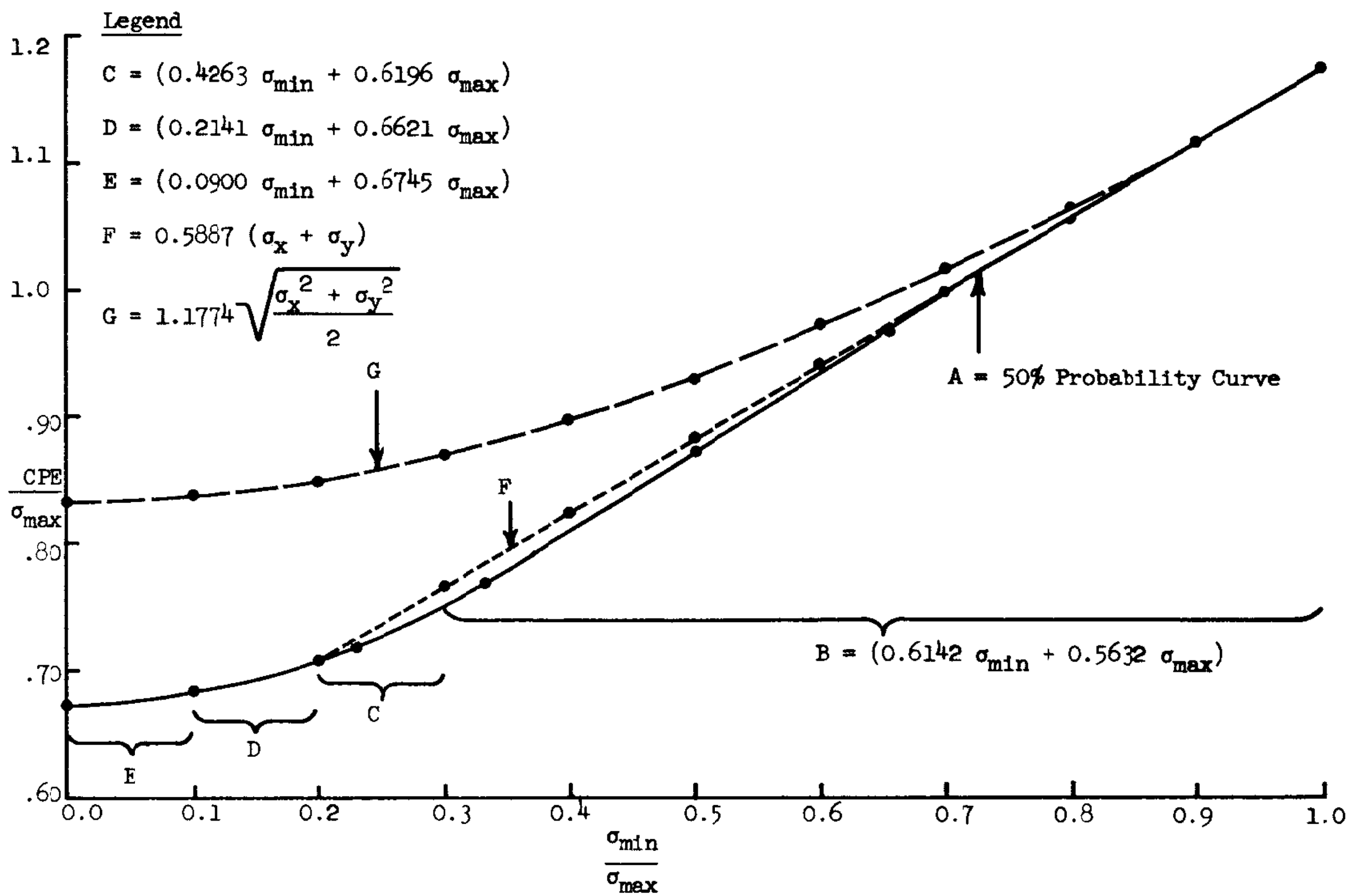


Figure 9

MSPE Probability Curve

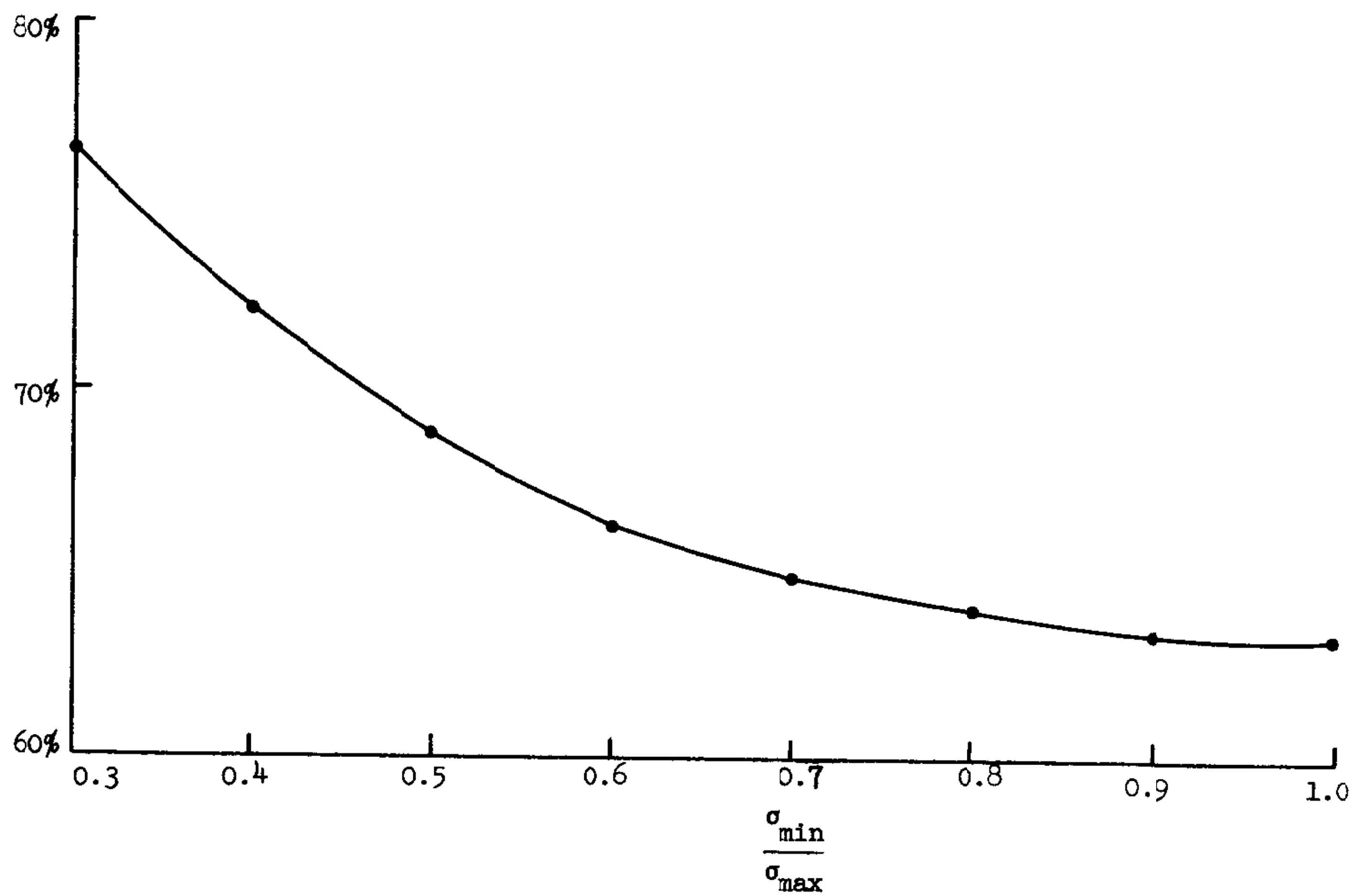


Figure 10

Graph of Conversion Factors
For 39.35% to 50% Probability

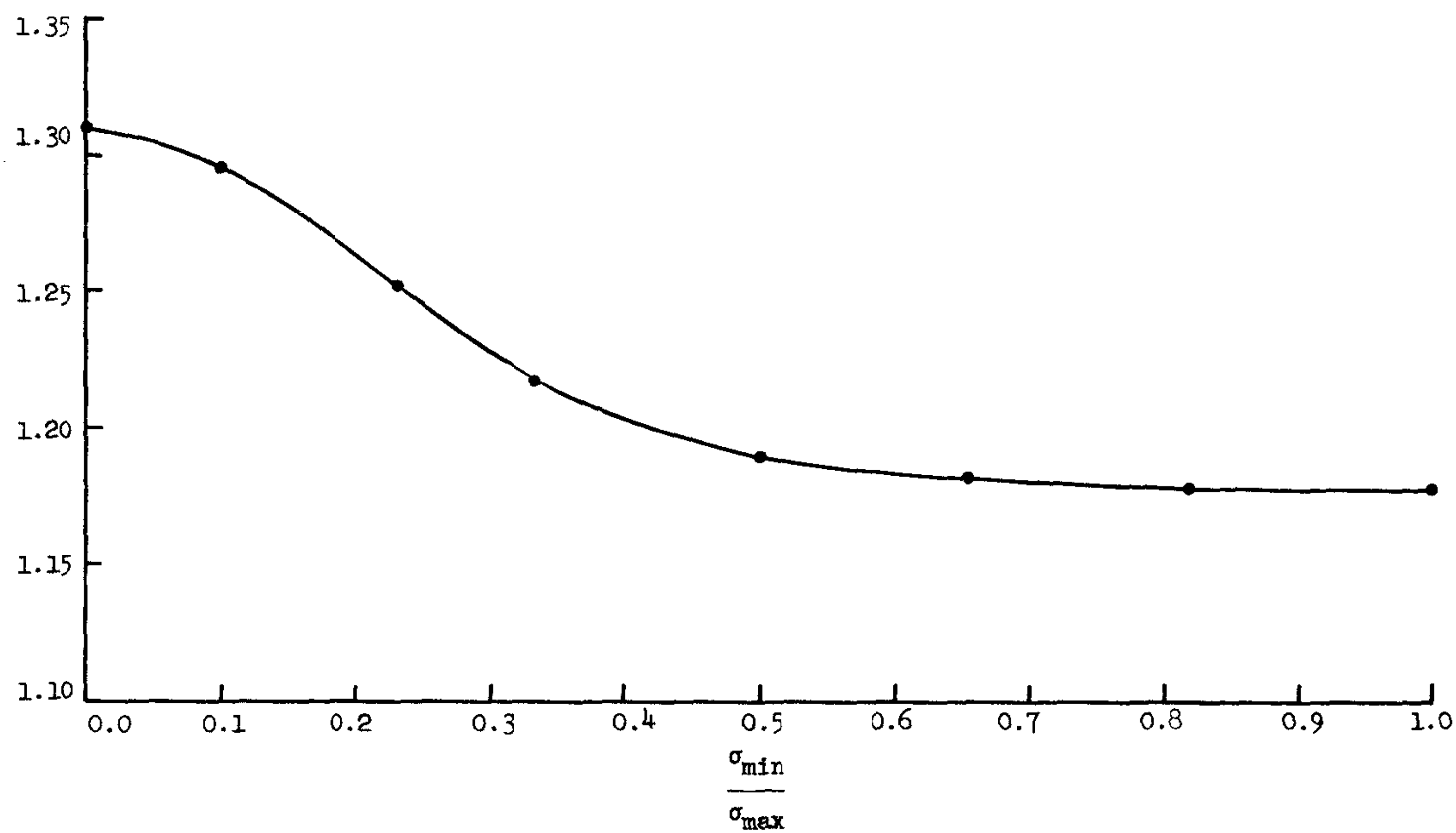
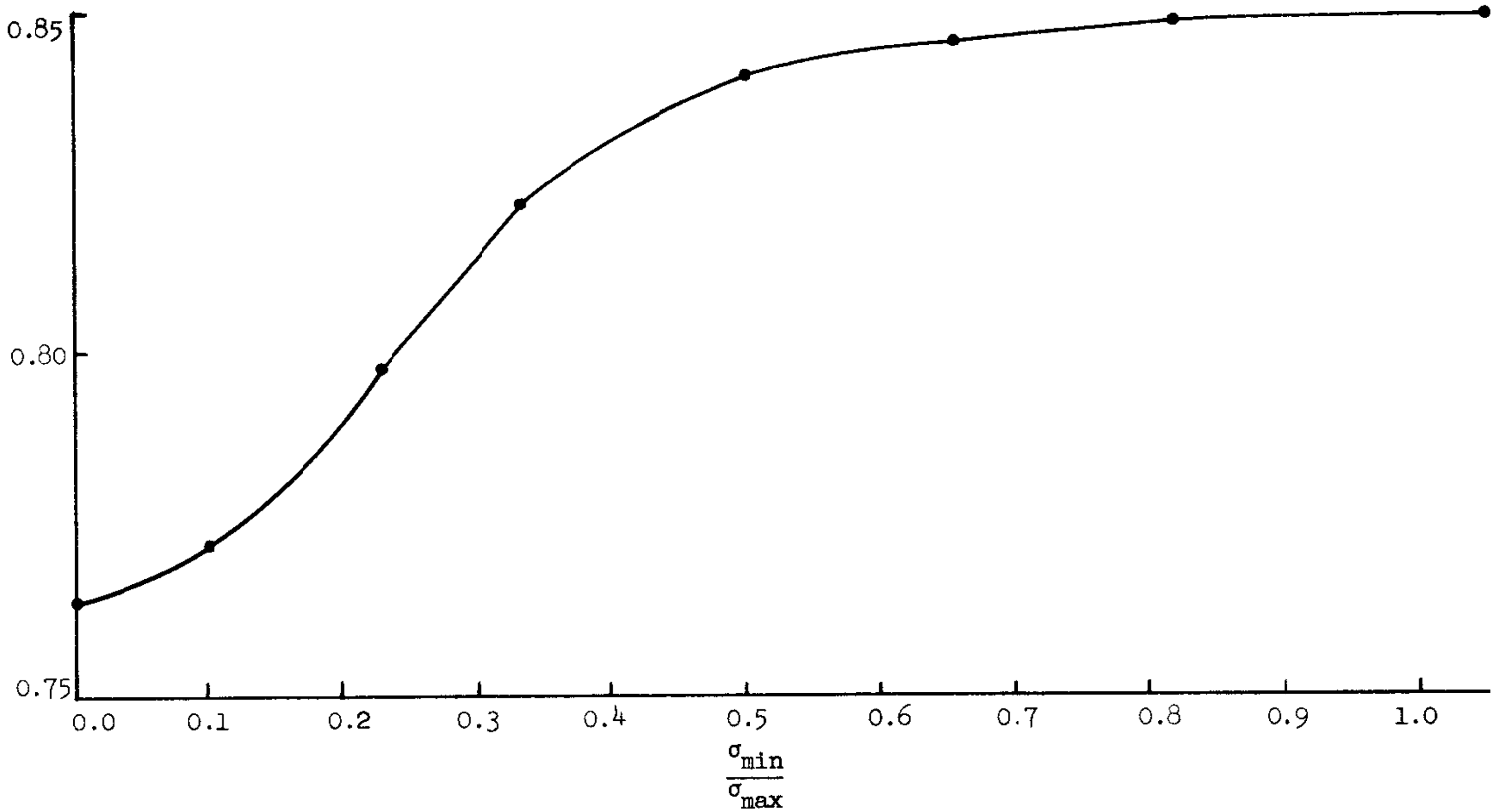


Figure 11

Graph of Conversion Factors
For 50.00% to 39.35% Probability



2.3.3. Circular Conversion Factors. (ref. 20, 27) The relationships of the circular standard error to other circular precision indexes are summarized in Table VI. Conversion factors (Table VII) computed from these relationships convert a circular error at a given probability to a circular error at another probability. When a circular error distribution is substituted for an elliptical distribution, the circular conversion factors are retained.

Table VI

Summary of Circular Precision Indexes

Symbol	Probability	Derivation
σ_c	.3935	1.0000 σ_c
CPE, CEP	.5000	1.1774 σ_c
MSPE	.6321	1.4142 σ_c
CMAS	.9000	2.1460 σ_c
3.5 σ_c	.9978	3.5000 σ_c

Table VII

Circular Error Conversion Factors

From \ To	39.35%	50.00%	63.21%	90.00%	99.78%
39.35%	1.0000	1.1774	1.4142	2.1460	3.5000
50.00%	0.8493	1.0000	1.2011	1.8227	2.9726
63.21%	0.7071	0.8325	1.0000	1.5174	2.4749
90.00%	0.4660	0.5486	0.6590	1.0000	1.6309
99.78%	0.2857	0.3364	0.4040	0.6131	1.0000

2.3.4. Propagation of Circular Errors. (ref. 5, 29) A two-dimensional quantity derived from a number of independent variables has a circular error resulting from the errors in each variable. The total circular error is determined by propagating the linear components in each of the two dimensions by methods described in Section 1.8., and computing the circular form by the procedure shown in Section 2.3.2. For example, the total circular error of a quantity C_T, derived from $C_T = C_1 + C_2 + \ldots C_n$, is found by:

$$\sigma_{x_T} = \sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2 + \ldots \sigma_{x_n}^2}$$

$$\sigma_{y_T} = \sqrt{\sigma_{y_1}^2 + \sigma_{y_2}^2 + \ldots \sigma_{y_n}^2}$$

$$\sigma_{c_T} = 0.5000\ (\sigma_{x_T} + \sigma_{y_T}) \qquad (2\text{-}27)$$

An alternate approximate propagation method combines the circular error of each independent variable directly, thus:

$$\sigma_{c_T} = \sqrt{\sigma_{c_1}^2 + \sigma_{c_2}^2 + \ldots \sigma_{c_n}^2} \qquad (2\text{-}28)$$

Precision indexes other than the standard error may be used; however, the index must be consistent throughout the computations.

3. THREE-DIMENSIONAL (ELLIPSOIDAL, SPHERICAL) ERRORS

3.1. Introduction. A three-dimensional error is the error in a quantity defined by three random variables. Expanding on the example in Section 2.1., a point is referred to X, Y, and Z axes which establish the spatial position of the point. When random and independent, the errors x, y, and z each have a linear probability distribution. The three-dimensional probability density function is expressed by:

$$p(x,y,z) = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_x \sigma_y \sigma_z} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right]} \quad (3\text{-}1)$$

Similar to Section 2.1., the probability density function can be written:

$$W^2 = \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2} \quad (3\text{-}2)$$

where:

$$W^2 = -2 \ln \left[p(x,y,z)\ \sigma_x\ \sigma_y\ \sigma_z\ (2\pi)^{\frac{3}{2}}\right]$$

For values of the constant W^2 from $0 \rightarrow \infty$, the density function defines a family of ellipsoids of equal probability density.

3.2. Ellipsoidal Errors. (ref. 15, 20) The probability of an error ellipsoid is given by the probability distribution function:

$$P(s) = \sqrt{\frac{2}{\pi}} \int_0^W t^2 e^{-\frac{1}{2}t^2}\, dt \quad (3\text{-}3)$$

where: s = the radial error; $s = \sqrt{x^2 + y^2 + z^2}$

$$t = \frac{s}{\sigma_{rs}^{\frac{3}{2}}}$$

σ_{rs} = standard error of the radial error "s"

The solution of equation (3-3) for W yields the values given in Table VIII.

Table VIII

Values for the Constant W

Probability	W
19.9%	1.000
50	1.538
60.8	1.732
90	2.500
99	3.368
99.89	4.000

3.3. Spherical Errors.

3.3.1. Spherical Probability Distribution Function. (ref.20)

When $\sigma_x = \sigma_y = \sigma_z = \sigma_{rs} \equiv \sigma_s$, equation (3-1) becomes the spherical probability density function:

$$p(s) = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_s^3} e^{-\frac{s^2}{2\sigma_s^2}} \qquad (3\text{-}4)$$

where: σ_s = spherical standard error

Integrating p(s) from s = 0 to s = S, equation (3-4) becomes the spherical probability distribution function:[1]

$$P(S) = \sqrt{\frac{2}{\pi}}\left[\left(\frac{S}{\sigma_s}\right) - e^{-\frac{S^2}{2\sigma_s^2}} + \int_0^S \frac{e}{s}^{-\frac{s^2}{2\sigma_s^2}} ds\right] \qquad (3\text{-}5)$$

where: S = radius of the probability sphere

Equation (3-5) can be solved by an approximation formula (ref. 11, 13):

$$P(S) \sim \sqrt{\frac{2}{\pi}}\left[1.253 - Ce^{-\frac{C^2}{2}} - \frac{e^{-\frac{C^2}{2}}}{C + 0.8\,e^{-0.4C}}\right] \qquad (3\text{-}6)$$

where: $C = \frac{S}{\sigma_s}$

3.3.2. Spherical Precision Indexes. (ref. 20, 27) A spherical error distribution is represented by indexes similar to those in Sections 1.6. and 2.3.2. Preferred spherical precision indexes include: (1) the spherical standard error (σ_s), (2) the spherical probable error (SPE), (3) the spherical accuracy standard (SAS), and (4) the spherical near-certainty error, four sigma ($4\sigma_s$). The mean radial spherical error (MRSE), an index which has been used at ACIC, is not recommended because the probability represented varies when σ_x, σ_y, and σ_z are not equal.

---

[1] See Appendix E.

The probability of an error sphere of radius equal to the spherical standard error is computed by equation (3-6) for the condition $C = \frac{S}{\sigma_S} = 1$ as follows:

$$\sqrt{\frac{2}{\pi}} = 0.7978846$$

$$e^{-\frac{1}{2}} = 0.60653$$

$$e^{-0.4} = 0.67032$$

$$0.8e^{-0.4} = 0.53626$$

$$P(S) \sim 0.79788\ (1.253 - 0.6065 - 0.3948)$$

$$\therefore P(S) \sim 0.20 \text{ or } 20\%\ ^{1} \qquad (3\text{-}7)$$

For a truly spherical distribution, the linear standard errors are equal and identical to the spherical standard error ($\sigma_x = \sigma_y = \sigma_z \equiv \sigma_s$). When σ_x, σ_y, and σ_z are not equal, the spherical standard error is approximated by:

$$\sigma_s \sim 0.3333\ (\sigma_x + \sigma_y + \sigma_z) \qquad (3\text{-}8)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.35

The substitution of a spherical form for an ellipsoidal distribution is not recommended when the $\sigma_{min}/\sigma_{max}$ ratio is less than 0.35.

The following alternate method of approximating σ_s is not recommended because of limited applicability:[2]

---

[1] A more accurate value is determined by an expansion in series to be 19.9% probability.

[2] Figure 12 compares curves computed from equations (3-8) and (3-9).

$$\sigma_s \sim \sqrt{\frac{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}{3}} \qquad (3\text{-}9)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.9

The <u>spherical probable error</u> is defined as the magnitude of the spherical radius S when the function P(S) = 0.5 or 50%. Expressed in the form $S = C\ \sigma_s$, the spherical probable error is computed by:

$$SPE = 1.5382\ \sigma_s \qquad (3\text{-}10)$$

The P(R) function for two-dimensional errors is solved by the use of Grad and Solomon's tables.[1] Expanding this method into the spherical distribution, the radius S for a 50% probability sphere ($S_{50\%}$) was computed in terms of σ_{max} for ratios of $\sigma_{min}/\sigma_{max}$ and $\sigma_{mid}/\sigma_{max}$ and tabulated in Table IX.[2] Utilizing these values, an approximation of the spherical probable error can be computed:[3]

$$SPE \sim 0.5127\ (\sigma_x + \sigma_y + \sigma_z) \qquad (3\text{-}11)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.35

The <u>mean radial spherical error</u> is the radius of the error sphere equal to $1.732\ \sigma_s$, or $\sqrt{3}\ \sigma_s$, in a truly spherical distribution. When $\sigma_x \neq \sigma_y \neq \sigma_z$, the MRSE is computed by:

---

[1] See Appendix D.

[2] where: σ_{min} = the minimum sigma, or smallest standard error of the three,
σ_{max} = the maximum sigma, and
σ_{mid} = the middle sigma.

[3] Note that $1.5382\ [0.3333\ (\sigma_x + \sigma_y + \sigma_z)] = 0.5127\ (\sigma_x + \sigma_y + \sigma_z)$.

$$\text{MRSE} \sim \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \qquad (3\text{-}12)$$

when $\sigma_{min}/\sigma_{max}$ is between 1.0 and 0.9

The probabilities represented by the MRSE are computed by equation (3-6).[1] Because of the variation in probability,[2] the MRSE is not recommended for use as a precision index.

The spherical accuracy standard is defined as the magnitude of the spherical radius S when the function P(S) = 0.9 or 90%. Expressed in the form $S = C\ \sigma_S$, the spherical accuracy standard is computed by:

$$\text{SAS} = 2.500\ \sigma_S \qquad (3\text{-}13)$$

The four sigma error, representing a spherical probability of 99.89%, approaches near-certainty in a spherical distribution and has a magnitude four times that of the spherical standard error.

---

[1] Illustrated in Figure 13.

[2] When $\sigma_x = \sigma_y = \sigma_z$, the probability is 60.82%; when $\sigma_x = 10$, $\sigma_y = 3$, and $\sigma_z = 6$, the probability is 69.36%.

Table IX

Solution of P(S) Function for P(S) = 50.00%

$\frac{\sigma_{mid}}{\sigma_{max}}$	$\frac{\sigma_{min}}{\sigma_{max}}$	SPE ~ $S_{50\%}$	SPE ~ 0.5127 ($\sigma_x + \sigma_y + \sigma_z$) Letting $\sigma_{max} = 1$
0.866	0.866	1.4016 σ_{max}	1.4007
1.0	0.707	1.3892 σ_{max}	1.3879
0.775	0.632	1.2341 σ_{max}	1.2341
0.577	0.577	1.1016 σ_{max}	1.1044
0.894	0.447	1.2104 σ_{max}	1.2002
0.707	0.408	1.0894 σ_{max}	1.0844
0.535	0.378	0.9791 σ_{max}	0.9808
0.354	0.354	0.8689 σ_{max}	0.8757

Figure 12

Comparison of Spherical Standard Error Approximation Methods

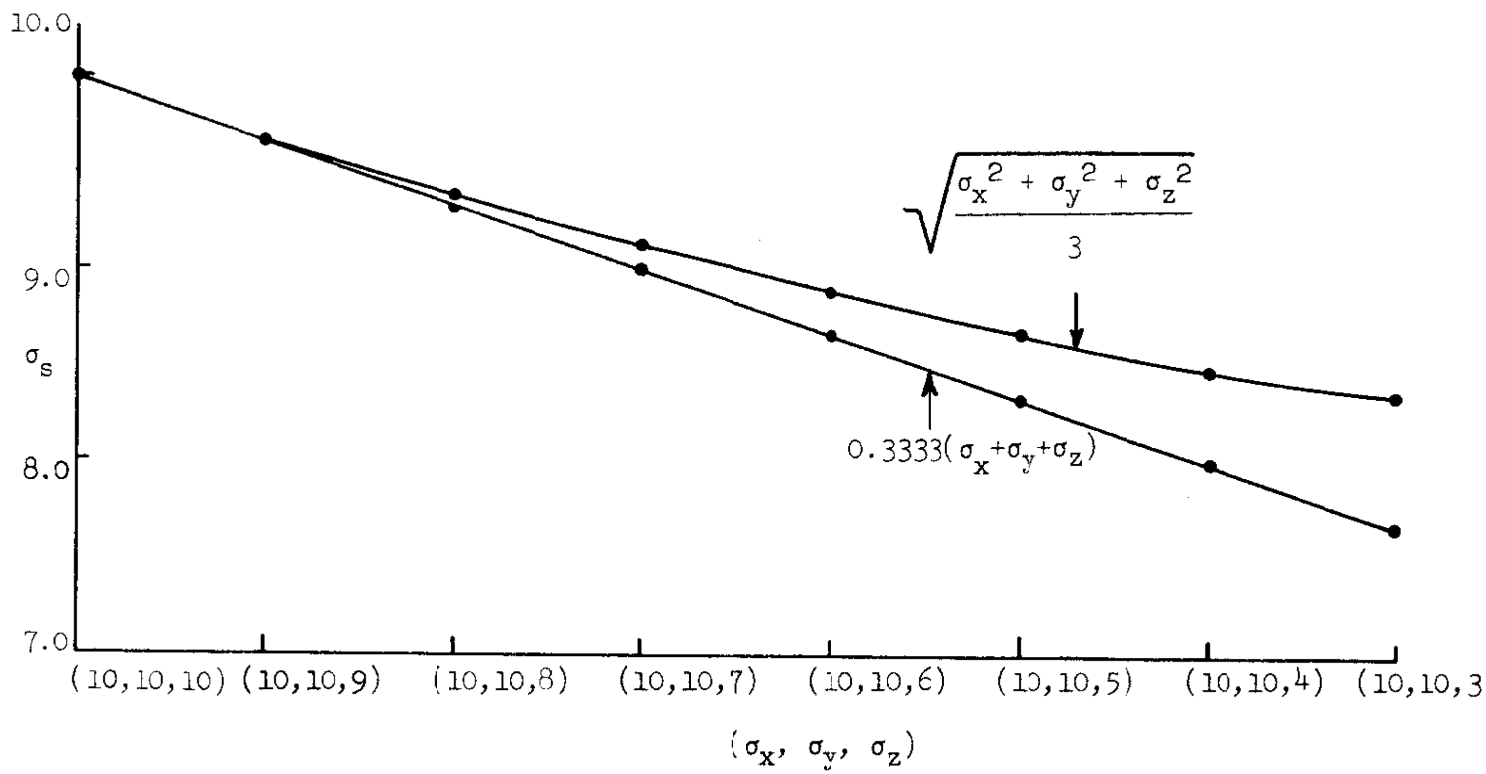
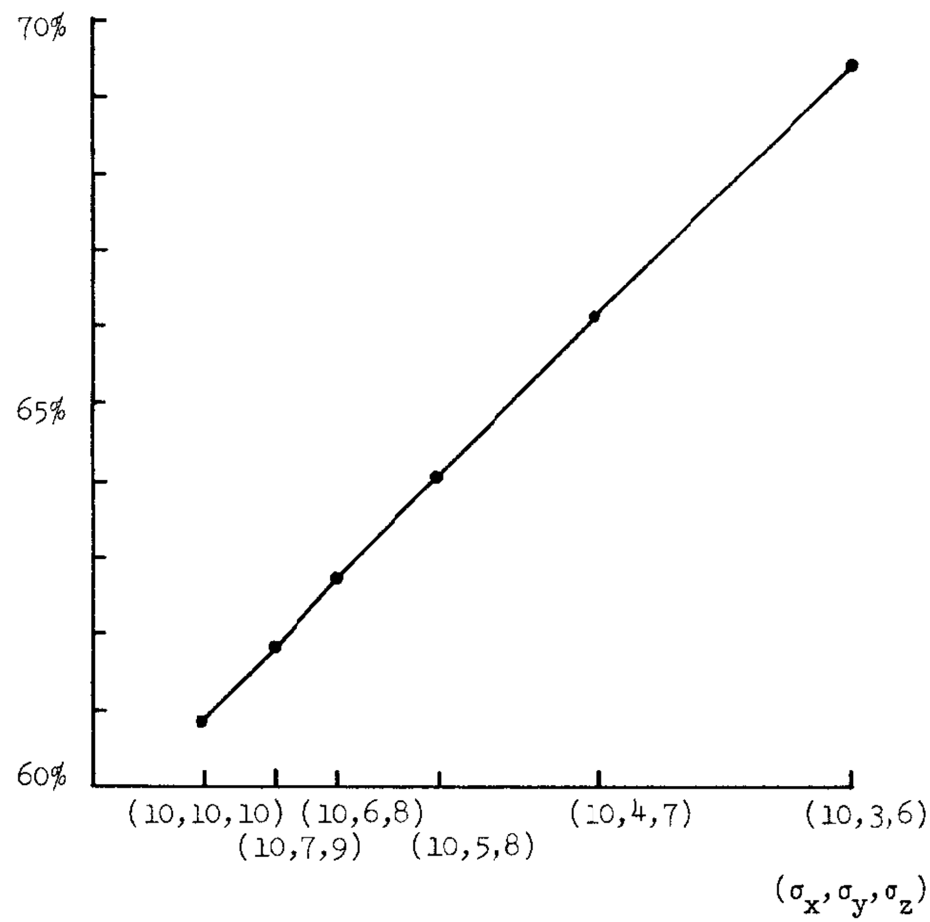


Figure 13

MRSE Probability Curve



3.3.3. Spherical Conversion Factors. (ref. 20, 27) The relationships of the spherical standard error to other spherical precision indexes are summarized in Table X. Conversion factors (Table XI) computed from these relationships convert a spherical error at a given probability to a spherical error at another probability.

Table X

Summary of Spherical Precision Indexes

Symbol	Probability	Derivation
σ_s	.199	1.000 σ_s
SPE	.50	1.538 σ_s
MRSE	.608	1.732 σ_s
SAS	.90	2.500 σ_s
4 σ_s	.9989	4.000 σ_s

Table XI

Spherical Error Conversion Factors

From \ To	19.9%	50%	60.8%	90%	99.89%
19.9%	1.000	1.538	1.732	2.500	4.000
50%	0.650	1.000	1.126	1.625	2.600
60.8%	0.577	0.888	1.000	1.443	2.309
90%	0.400	0.615	0.693	1.000	1.600
99.89%	0.250	0.385	0.433	0.625	1.000

3.3.4. Propagation of Spherical Errors. (ref. 5, 29) A three-dimensional quantity derived from a number of independent variables has a spherical error resulting from the errors in each variable. The total spherical error is determined by propagating the linear components in each of the three dimensions by methods described in Section 1.8., and computing the spherical form by the procedure shown in Section 3.3.2. For example, the total spherical error of a quantity S_T, derived from $S_T = S_1 + S_2 + \ldots S_n$ is found by:

$$\sigma_{x_T} = \sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2 + \ldots \sigma_{x_n}^2}$$

$$\sigma_{y_T} = \sqrt{\sigma_{y_1}^2 + \sigma_{y_2}^2 + \ldots \sigma_{y_n}^2}$$

$$\sigma_{z_T} = \sqrt{\sigma_{z_1}^2 + \sigma_{z_2}^2 + \ldots \sigma_{z_n}^2}$$

$$\sigma_{s_T} = 0.3333 \left(\sigma_{x_T} + \sigma_{y_T} + \sigma_{z_T}\right) \qquad (3\text{-}14)$$

An alternate approximate propagation method combines the spherical error of each independent variable directly, thus:

$$\sigma_{s_T} = \sqrt{\sigma_{s_1}^2 + \sigma_{s_2}^2 + \ldots \sigma_{s_n}^2} \qquad (3\text{-}15)$$

Precision indexes other than the standard error may be used; however, the index must be consistent throughout the computations.

4. APPLICATION OF ERROR THEORY TO POSITIONAL INFORMATION

4.1. Positional Errors. By the use of error theory in the evaluation of ACIC positional information, it is possible to establish a meaningful accuracy statement subject to uniform interpretation. To provide a logical and acceptable basis for computation and comparison, positional errors are assumed to follow a normal distribution. The assumption is valid because positional error components generally follow a normal distribution pattern when sufficient data is available.

The statistical treatment of errors is applied to measurable quantities found in the sources of positioning information. The differences between the surveyed coordinates of ground control and the scaled coordinates of the same control symbolized on maps are considered to be the errors in the geodetic base of the map. Analysis of the linear components — latitude and longitude or grid Northing and Easting — provides a two-dimensional expression of the accuracy of the geodetic base. When all the linear standard errors occurring during map construction are combined and converted to a circular distribution, the final map accuracy statement is expressed in terms of circular errors.

Among the positioning errors in maps, there are often those which are not measurable and which must be estimated by empirical methods. When this is necessary, an additional assumption must be made to the effect that such data is compatible with computed data and that empirically derived error data will also follow the theoretical error distribution.

Various types of points require different parameters to establish precise positions. These have been discussed as one, two, or three-dimensional coordinates. For example, a vertical position (elevation) requires only a one-dimensional coordinate — the height of the point above a reference datum; a geodetic position is expressed by two-dimensional coordinates — latitude and longitude referenced to a specific datum; and spatial positions require three-dimensional coordinates such as the x, y, z coordinates in a rectangular system. The errors accumulated in the process of determining the various positions must be evaluated in the same dimensions required to express the position. Errors for vertical positioning can be assumed to follow a normal linear distribution; those for a geodetic position — a circular distribution; and the errors for a spatial point can be assumed to follow a normal spherical distribution.

4.2. The Accuracy Statement. Two major groups of data fall within Air Force positioning requirements: (1) maps, charts, and other graphics; and (2) specific points. By the use of error theory, a horizontal accuracy evaluation of the graphic as a whole can be obtained, i.e., a specified probability that the true errors in well-defined planimetry will not exceed the given quantity. Map accuracy can also be interpreted as percentage — the percentage of well-defined points which will not contain errors exceeding the given magnitude. Similarly, vertical accuracy is stated as a given probability that the linear errors in vertical position are not likely to exceed a specified value.

The accuracy of a specific point is expressed also by a statement of probability and error magnitude. The accuracy statement does not mean that the error in position is exactly the value shown, rather it expresses the probability that the true error in position will not be larger than the error given.

Positional error should be expressed by precision indexes which immediately identify the form and probability represented by a given error. For example, let the circular probable error (CPE) of a geodetic position equal 100 feet. Then the form is circular. The magnitude 100 feet and the probability (50% by definition of CPE) are derived from a statistical treatment of known or estimated error components comprising the total positional error. The statement infers a 50-50 chance that the geodetic position in question does not vary more than 100 feet from the true geodetic position. When the error magnitude is increased by a statistical factor, greater probability is achieved. Multiplying 100 feet by 1.8227 yields a 90% probability that the positional error will not exceed 182 feet.

Errors in different forms are more easily understood when precision indexes common to linear, circular, and spherical error distributions are used. Precision indexes suitable for expressing positional error include (1) the linear, circular, and spherical standard errors representing 68.27%, 39.35%, and 19.9% probabilities, respectively, (2) the linear probable error, circular probable error, and spherical probable error representing 50% probability in each distribution, (3) the map accuracy standard, circular map accuracy standard, and spherical accuracy standard representing a 90% probability level, and (4) a probability level approaching near-certainty for each distribution which the positional error is theoretically unlikely to exceed; (a) three sigma (linear, 99.73%), (b) three-five sigma (circular, 99.78%), and (c) four sigma (spherical, 99.89%). Since error values are easily converted from one precision index to another in the same distribution, the use of any index is largely a matter of choice. However, in presenting positional information, the positional error is best expressed by either the 50% or 90% probability precision index or both.

4.3. Summary of Formulas and Conversion Factors.

Linear Error Formulas			
Precision Index	Symbol	Percentage Probability	Formula
Standard Error	σ[1]	68.27%	$\sigma_x = \sqrt{\frac{\Sigma (X_i - \overline{X})^2}{n-1}} = \sqrt{\frac{\Sigma x^2}{n-1}}$ where: X_i = a measured value of the quantity X; $X_1, X_2 \ldots X_n$ $\overline{X}$ = the most probable value (arithmetic mean) of X $\overline{X} = \frac{\Sigma X_i}{n}$ x = the error; $x = X_i - \overline{X}$ n = number of measurements
Probable Error	PE	50%	PE = $0.6745\ \sigma_x$
Map Accuracy Standard	MAS	90%	MAS = $1.6449\ \sigma_x$
Near-Certainty Error (Three sigma)	3 σ	99.73%	$3.0000\ \sigma_x$

Linear Error Conversion Factors				
From \ To	50%	68.27%	90%	99.73%
50%	1.0000	1.4826	2.4387	4.4475
68.27	0.6745	1.0000	1.6449	3.0000
90	0.4101	0.6080	1.0000	1.8239
99.73	0.2248	0.3333	0.5483	1.0000

[1]Subscripts denote the standard error computed from a sample (σ_x, σ_y, σ_z).

Circular Error Formulas			
Precision Index	Symbol	Percentage Probability	Formula
Circular Standard Error	σ_c	39.35%	$\sigma_c = 0.5000\ (\sigma_x + \sigma_y)$ when $\sigma_{min}/\sigma_{max} \geq 0.2$ [1]
Circular Probable Error	CPE, CEP	50%	CPE $= 1.1774\ \sigma_c$ CPE $= 0.5887\ (\sigma_x + \sigma_y)$ when $\sigma_{min}/\sigma_{max} \geq 0.2$ CPE $\sim (0.2141\ \sigma_{min} + 0.6621\ \sigma_{max})$ when $0.1 \leq \sigma_{min}/\sigma_{max} \leq 0.2$ [2] CPE $\sim (0.0900\ \sigma_{min} + 0.6745\ \sigma_{max})$ when $0.0 \leq \sigma_{min}/\sigma_{max} \leq 0.1$ [2]
Circular Map Accuracy Standard	CMAS	90%	CMAS $= 2.1460\ \sigma_c$ CMAS $= 1.0730\ (\sigma_x + \sigma_y)$ when $\sigma_{min}/\sigma_{max} \geq 0.2$
Circular Near-Certainty Error (Three-five sigma)	$3.5\sigma_c$	99.78%	$3.5000\ \sigma_c$

Circular Error Conversion Factors					
From \ To	39.35%	50%	63%	90%	99.78%
39.35%	1.0000	1.1774	1.4142	2.1460	3.5000
50	0.8493	1.0000	1.2011	1.8227	2.9726
63	0.7071	0.8325	1.0000	1.5174	2.4749
90	0.4660	0.5486	0.6590	1.0000	1.6309
99.78	0.2857	0.3364	0.4040	0.6131	1.0000

[1] Where σ_{min} is the minimum or smaller linear standard error of the two.

[2] A circular error concept is not recommended for $\sigma_{min}/\sigma_{max}$ ratios less than 0.2. However, a near-linear 50% probability error may be computed to represent a CPE for lower ratios when a comparison of circular errors derived from different sources is required.

Spherical Error Formulas			
Precision Index	Symbol	Percentage Probability	Formula
Spherical Standard Error	σ_s	19.9%	$\sigma_s = 0.3333(\sigma_x + \sigma_y + \sigma_z)$ when $\sigma_{min}/\sigma_{max} \geq 0.35$[1]
Spherical Probable Error	SPE	50%	SPE $= 1.5382\ \sigma_s$ SPE $= 0.5127\ (\sigma_x + \sigma_y + \sigma_z)$ when $\sigma_{min}/\sigma_{max} \geq 0.35$
Spherical Accuracy Standard	SAS	90%	SAS $= 2.5003\ \sigma_s$ SAS $= 0.8333\ (\sigma_x + \sigma_y + \sigma_z)$ when $\sigma_{min}/\sigma_{max} \geq 0.35$
Spherical Near-Certainty Error (Four sigma)	$4\ \sigma_s$	99.89%	$4.0000\ \sigma_s$

Spherical Error Conversion Factors					
From \ To	19.9%	50%	61%	90%	99.89%
19.9%	1.000	1.538	1.732	2.500	4.000
50	0.650	1.000	1.126	1.625	2.600
61	0.577	0.888	1.000	1.443	2.309
90	0.400	0.615	0.693	1.000	1.600
99.89	0.250	0.385	0.433	0.625	1.000

[1] A spherical concept is not recommended when $\sigma_{min}/\sigma_{max}$ is less than 0.35.

Appendix A

PERCENTAGE PROBABILITY FOR STANDARD ERROR INCREMENTS[1]

The following table presents the increments of linear (σ_x), circular (σ_c), and spherical (σ_s) standard errors for intervals of one percent probability. Percentage levels corresponding to precision indexes are underlined.

Factors for converting the error at one percentage probability to another within the same distribution are derived by dividing the standard error increment of the new percentage probability by the standard error increment of the given percentage probability. An example is the conversion from the circular map accuracy standard (90%) to the circular probable error (50%):

$$\text{CPE} = 1.1774\ \sigma_c$$

$$\text{CMAS} = 2.1460\ \sigma_c$$

$$\text{CPE} = \frac{1.1774}{2.1460}\ \text{CMAS}$$

$$\therefore\ \text{CPE} = 0.5486\ \text{CMAS}$$

%	σ_x	σ_c	σ_s
00	0.0000	0.0000	0.0000
01	0.0125	0.1418	0.3389
02	0.0251	0.2010	0.4299

[1] Reference 27.

%	σ_x	σ_c	σ_s
03	0.0376	0.2468	0.4951
04	0.0502	0.2857	0.5479
05	0.0627	0.3203	0.5932
06	0.0753	0.3518	0.6334
07	0.0878	0.3810	0.6699
08	0.1004	0.4084	0.7035
09	0.1130	0.4343	0.7349
10	0.1257	0.4590	0.7644
11	0.1383	0.4828	0.7924
12	0.1510	0.5056	0.8192
13	0.1637	0.5278	0.8447
14	0.1764	0.5492	0.8694
15	0.1891	0.5701	0.8932
16	0.2019	0.5905	0.9162
17	0.2147	0.6105	0.9386
18	0.2275	0.6300	0.9605
19	0.2404	0.6492	0.9818
19.9			1.0000
20	0.2533	0.6680	1.0026
21	0.2663	0.6866	1.0230
22	0.2793	0.7049	1.0430
23	0.2924	0.7230	1.0627
24	0.3055	0.7409	1.0821
25	0.3186	0.7585	1.1012
26	0.3319	0.7760	1.1200
27	0.3451	0.7934	1.1386
28	0.3585	0.8106	1.1570
29	0.3719	0.8276	1.1751
30	0.3853	0.8446	1.1932
31	0.3989	0.8615	1.2110
32	0.4125	0.8783	1.2288
33	0.4261	0.8950	1.2464
34	0.4399	0.9116	1.2638
35	0.4538	0.9282	1.2812
36	0.4677	0.9448	1.2985
37	0.4817	0.9613	1.3158
38	0.4959	0.9778	1.3330
39	0.5101	0.9943	1.3501
39.35		1.0000	
40	0.5244	1.0108	1.3672
41	0.5388	1.0273	1.3842
42	0.5534	1.0438	1.4013
43	0.5681	1.0603	1.4183
44	0.5828	1.0769	1.4354
45	0.5978	1.0935	1.4524

%	σ_x	σ_c	σ_s
46	0.6128	1.1101	1.4695
47	0.6280	1.1268	1.4866
48	0.6433	1.1436	1.5037
49	0.6588	1.1605	1.5209
50	0.6745	1.1774	1.5382
51	0.6903	1.1944	1.5555
52	0.7063	1.2116	1.5729
53	0.7225	1.2288	1.5904
54	0.7388	1.2462	1.6080
55	0.7554	1.2637	1.6257
56	0.7722	1.2814	1.6436
57	0.7892	1.2992	1.6616
57.51	0.7979		
58	0.8064	1.3172	1.6797
59	0.8239	1.3354	1.6980
60	0.8416	1.3537	1.7164
60.82			1.7321
61	0.8596	1.3723	1.7351
62	0.8779	1.3911	1.7540
63	0.8965	1.4101	1.7730
63.21		1.4142	
64	0.9154	1.4294	1.7924
65	0.9346	1.4490	1.8119
66	0.9542	1.4689	1.8318
67	0.9741	1.4891	1.8519
68	0.9945	1.5096	1.8724
68.27	1.0000		
69	1.0152	1.5305	1.8932
70	1.0364	1.5518	1.9144
71	1.0581	1.5735	1.9360
72	1.0803	1.5956	1.9580
73	1.1031	1.6182	1.9804
74	1.1264	1.6414	2.0034
75	1.1503	1.6651	2.0269
76	1.1750	1.6894	2.0510
77	1.2004	1.7145	2.0757
78	1.2265	1.7402	2.1012
79	1.2536	1.7667	2.1274
80	1.2816	1.7941	2.1544
81	1.3106	1.8225	2.1825
82	1.3408	1.8519	2.2114
83	1.3722	1.8825	2.2416
84	1.4051	1.9145	2.2730
85	1.4395	1.9479	2.3059

%	σ_x	σ_c	σ_s
86	1.4758	1.9830	2.3404
87	1.5141	2.0200	2.3767
88	1.5548	2.0593	2.4153
89	1.5982	2.1011	2.4563
90	1.6449	2.1460	2.5003
91	1.6954	2.1945	2.5478
92	1.7507	2.2475	2.5998
93	1.8119	2.3062	2.6571
94	1.8808	2.3721	2.7216
95	1.9600	2.4477	2.7955
96	2.0537	2.5373	2.8829
97	2.1701	2.6482	2.9912
98	2.3263	2.7971	3.1365
99	2.5758	3.0349	3.3683
99.73	3.0000		
99.78		3.5000	
99.89			4.0000
99.9	3.2905	3.7169	4.0345
99.99	3.8905	4.2919	4.6094

Appendix B

THE MOST PROBABLE VALUE

Since the true value of a measured quantity is never known, the most probable value of the quantity must be determined from the observed values. The following proof (ref. no. 5) will show that the arithmetic mean of the observed values is the most probable value of the quantity:

Symbols:

X = an unknown quantity

X_i = the observed values of the unknown quantity;

$$X_i = X_1, X_2, X_3 \dots X_n \qquad (1)$$

$\bar{X}$ = the arithmetic mean of the observed values;

$$\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}, \text{ or } n\bar{X} = \sum_{i=1}^{n} X_i \qquad (2)$$

x_i = the error in an observation;

$$x_i = X_i - \bar{X} \qquad (3)$$

Proof:

$$x_1 = X_1 - \bar{X}$$

$$x_2 = X_2 - \bar{X}$$

$$\dots\dots\dots\dots$$

$$x_n = X_n - \bar{X}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} X_i - n\bar{X}$$

From equation (2);

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} X_i = 0 \qquad (4)$$

This shows that the sum of the differences about the mean is zero, which was expected, but if equation (3) is squared and then summed:

$$x_1^2 = X_1^2 - 2X_1 \overline{X} + \overline{X}^2 \qquad (5)$$

$$x_2^2 = X_2^2 - 2X_2 \overline{X} + \overline{X}^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n^2 = X_n^2 - 2X_n \overline{X} + \overline{X}^2$$

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} X_i^2 - 2\overline{X} \sum_{i=1}^{n} X_i + n\overline{X}^2 \qquad (6)$$

The most probable value will be found when $\sum_{i=1}^{n} x_i^2 = 0$, or the most probable value of $\overline{X}$ will be that which makes $\sum_{i=1}^{n} x_i^2$ = a minimum.

In order to find this minimum, differentiate equation (6) with respect to $\overline{X}$ and equate to 0:

$$\frac{d}{d\overline{X}} \sum_{i=1}^{n} x_i^2 = -2 \sum_{i=1}^{n} X_i + 2 n \overline{X} = 0$$

$$\therefore \overline{X} = \sum_{i=1}^{n} \frac{X_i}{n} \qquad (7)$$

Equation (7) proves that the mean value $\overline{X}$ is the most probable value of a set of independent observations. Therefore, in the determination of the residual value it is correct to use the mean value for an approximation of the true value.

Appendix C

PROPAGATION OF ERRORS

A quantity f_i is computed from two measured quantities a and b, where f(a,b) denotes a function of a and b. The error Δf of f_i is affected by the errors in both a and b: Δa and Δb. Assuming a and b are independent, and the errors Δa, Δb are randomly distributed, the combined error Δf can be computed. (ref. nos. 5, 15)

Let:

$$f_1 = f\,(a_1, b_1)$$

$$f_2 = f\,(a_2, b_2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_n = f\,(a_n, b_n) \qquad (1)$$

The measured values of a and b may be averaged, obtaining the values $\bar{a}$ and $\bar{b}$. The most probable value of f is $\bar{f}$, (from appendix B), where:

$$\bar{f} = f\,(\bar{a}, \bar{b})$$

and:

$$\Delta f_i = f_i - \bar{f} \qquad (2)$$

In order to find the value of Δf_i, take the partial derivative of f_i:

$$\Delta f_i = \frac{\partial f_i}{\partial a_i}\,\Delta a_i + \frac{\partial f_i}{\partial b_i}\,\Delta b_i \qquad (3)$$

From Appendix B, $\sum_{i=1}^{n} \Delta f_i = 0$

Computing the sum of the squares of equation (3):

$$(\Delta f_1)^2 = \left(\frac{\partial f_1}{\partial a_1}\right)^2 \Delta a_1^2 + 2\left(\frac{\partial f_1}{\partial a_1}\right)\left(\frac{\partial f_1}{\partial b_1}\right) \Delta a_1 \, \Delta b_1 + \left(\frac{\partial f_1}{\partial b_1}\right)^2 \Delta b_1^2$$

$$(\Delta f_2)^2 = \left(\frac{\partial f_2}{\partial a_2}\right)^2 \Delta a_2^2 + 2\left(\frac{\partial f_2}{\partial a_2}\right)\left(\frac{\partial f_2}{\partial b_2}\right) \Delta a_2 \, \Delta b_2 + \left(\frac{\partial f_2}{\partial b_2}\right)^2 \Delta b_2^2$$

....

$$(\Delta f_n)^2 = \left(\frac{\partial f_n}{\partial a_n}\right)^2 \Delta a_n^2 + 2\left(\frac{\partial f_n}{\partial a_n}\right)\left(\frac{\partial f_n}{\partial b_n}\right) \Delta a_n \, \Delta b_n + \left(\frac{\partial f_n}{\partial b_n}\right)^2 \Delta b_n^2$$

Since:

$$\frac{\partial f_1}{\partial a_1} = \frac{\partial f_2}{\partial a_2} = \frac{\partial f_n}{\partial a_n} = \text{a constant;}$$

also:

$$\frac{\partial f_1}{\partial b_1} = \frac{\partial f_2}{\partial b_2} = \frac{\partial f_n}{\partial b_n} = \text{a constant:}$$

$$\sum_{i=1}^{n} \Delta f_i^2 = \left(\frac{\partial f_i}{\partial a_i}\right)^2 \sum_{i=1}^{n} \Delta a_i^2 + 2\left(\frac{\partial f_i}{\partial a_i}\right)\left(\frac{\partial f_i}{\partial b_i}\right) \sum_{i=1}^{n} \Delta a_i \, \Delta b_i$$

$$+ \left(\frac{\partial f_i}{\partial b_i}\right)^2 \sum_{i=1}^{n} \Delta b_i^2 \qquad (4)$$

Dividing through by n:

$$\sum_{i=1}^{n} \frac{\Delta f_i^{\,2}}{n} = \left(\frac{\partial f_i}{\partial a_i}\right)^2 \sum_{i=1}^{n} \frac{\Delta a_i^{\,2}}{n} + 2 \left(\frac{\partial f_i}{\partial a_i}\right) \left(\frac{\partial f_i}{\partial b_i}\right) \sum_{i=1}^{n} \frac{\Delta a_i \; \Delta b_i}{n}$$

$$+ \left(\frac{\partial f_i}{\partial b_i}\right)^2 \sum_{i=1}^{n} \frac{\Delta b_i^{\,2}}{n} \qquad (5)$$

By definition:

$$\sum_{i=1}^{n} \frac{\Delta f^2}{n} = \sigma_{f_i}^{\;2} \; ; \sum_{i=1}^{n} \frac{\Delta a^2}{n} = \sigma_a^{\;2} \; ; \sum_{i=1}^{n} \frac{\Delta b^2}{n} = \sigma_b^{\;2} \qquad (6)$$

Since a and b are independent:

$$\left(\frac{\partial f_i}{\partial a_i}\right) \left(\frac{\partial f_i}{\partial b_i}\right) \sum_{i=1}^{n} \frac{\Delta a \; \Delta b}{n} = 0 \qquad (7)$$

Therefore:

$$\sigma_{f_i} = \sqrt{\left(\frac{\partial f_i}{\partial a_i}\right)^2 \sigma_a^{\;2} + \left(\frac{\partial f_i}{\partial b_i}\right)^2 \sigma_b^{\;2}} \qquad (8)$$

Equation (8) is the general form for the propagation of independent errors, and can be expanded to cover any number of quantities (a, b, c, d, ...). It is imperative that each element represent the same precision index in the equation.

Special Rules for Error Propagation

Rule 1. Sum and Difference: $f = (a + b + \ \ldots)$ or $f = (a - b - \ \ldots)$

$$\frac{\partial f}{\partial a} = 1, \quad \frac{\partial f}{\partial b} = 1 \qquad (9)$$

Placing (9) in the general equation (8):

$$\sigma_f = \sqrt{\sigma_a^2 + \sigma_b^2 + \ldots} \qquad (10)$$

The absolute standard error of a quantity computed from the sum or difference of measured quantities is equal to the square root of the sum of the squared standard errors of the measured quantities. This is the form most frequently encountered.

Rule 2. Product of Factors Raised To Various Powers: $f = a^m b^q$

$$\frac{\partial f}{\partial a} = m a^{m-1} b^q \quad \text{and} \quad \frac{\partial f}{\partial b} = a^m q b^{q-1} \qquad (11)$$

Placing (11) into equation (8):

$$\sigma_f = \sqrt{m^2 a^{2m-2} b^{2q} \sigma_a^2 + a^{2m} q^2 b^{2q-2} \sigma_b^2} \qquad (12)$$

Dividing through by $f = \sqrt{a^{2m} b^{2q}}$:

$$\frac{\sigma_f}{f} = \sqrt{\frac{m^2 a^{2m-2} b^{2q} \sigma_a^2}{a^{2m} b^{2q}} + \frac{a^{2m} q^2 b^{2q-2} \sigma_b^2}{a^{2m} b^{2q}}}$$

$$\frac{\sigma_f}{f} = \sqrt{m^2 \left(\frac{\sigma_a}{a}\right)^2 + q^2 \left(\frac{\sigma_b}{b}\right)^2} \qquad (13)$$

Rule 3. Simple Product or Quotient: From the preceeding rule,

$$f = a^m b^q, \text{ let } m = 1, q = \pm 1.$$

Then, $f = ab$, or $f = a/b$.

From Equation (13):

$$\frac{\sigma_f}{f} = \sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2} \qquad (14)$$

where σ_f/f is the fractional standard error.

Appendix D

DERIVATION AND SOLUTION OF THE TWO-DIMENSIONAL PROBABILITY DISTRIBUTION FUNCTION

1. Derivation. (ref. no. 24) The probability density functions of the independent errors "x" and "y" are:

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}}, \text{ and } p(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_y^2}}$$

Using Rule 4, Section 1.3.:

$$p(x,y) = \frac{1}{2\pi \sigma_x \sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

$$P(x,y) = \frac{1}{2\pi \sigma_x \sigma_y} \int_x \int_y e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]} dx\, dy \qquad (1)$$

Using polar coordinates:

$$x^2 = r^2 \cos^2 \theta$$

$$y^2 = r^2 \sin^2 \theta$$

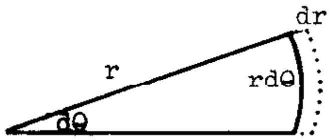
where r is the radial error and $r = \sqrt{x^2 + y^2}$

$$P(r) = P\left| r = \sqrt{x^2 + y^2} \leq R \right| = P \left| xy < R \right| \qquad (2)$$

where R is the radius of the probability circle.

The two-dimensional probability distribution function is:

$$P(R) = \frac{1}{2\pi\, \sigma_x\, \sigma_y} \int_{r=0}^{R} \int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}\left[\frac{\sin^2\theta}{\sigma_y^2} + \frac{\cos^2\theta}{\sigma_x^2}\right]} r\, dr\, d\theta$$



r dθ dr (small increment resulting from dx and dy)

Using identities: $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$P(R) = \frac{1}{2\pi\, \sigma_x\, \sigma_y} \int_{r=0}^{R} \int_{\theta=0}^{2\pi} e^{-\frac{r^2}{4}\left[\frac{1 - \cos 2\theta}{\sigma_y^2} + \frac{1 + \cos 2\theta}{\sigma_x^2}\right]} r\, dr\, d\theta$$

Rearranging terms:

$$P(R) = \frac{1}{2\pi\, \sigma_x\, \sigma_y} \int_{r=0}^{R} r e^{-\frac{r^2}{4}\left[\frac{1}{\sigma_y^2} + \frac{1}{\sigma_x^2}\right]} 4 \int_{\theta=0}^{\frac{\pi}{2}} e^{-\frac{r^2}{4}\left[\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right]\cos 2\theta} d\theta dr$$

Let $\phi = 2\theta$,

$d\phi = 2d\theta$:

Then:

$$P(R) = \frac{1}{2\pi\ \sigma_x\ \sigma_y} \int_{r=0}^{R} re^{-\frac{r^2}{4}\left[\frac{1}{\sigma_y^2} + \frac{1}{\sigma_x^2}\right]} 4 \int_{\phi=0}^{\pi} e^{-\frac{r^2}{4}\left[\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right]\cos\phi} \frac{d\phi}{2}\, dr$$

Rearranging terms:

$$P(R) = \frac{1}{\sigma_x\ \sigma_y} \int_{r=0}^{R} re^{-\frac{r^2}{4\sigma_y^2}\left[1 + \frac{\sigma_y^2}{\sigma_x^2}\right]} \left[\frac{1}{\pi} \int_{\phi=0}^{\pi} e^{-\frac{r^2}{4\sigma_y^2}\left[\frac{\sigma_y^2}{\sigma_x^2} - 1\right]\cos\phi}\, d\phi\right] dr$$

Let:

$$\left[\frac{1}{\pi} \int_{0}^{\pi} e^{-\frac{r^2}{4\sigma_y^2}\left[\frac{\sigma_y^2}{\sigma_x^2} - 1\right]\cos\phi}\, d\phi\right] = I_o\left[\frac{r^2}{4\sigma_y^2}\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right]$$

where I_o is a Bessel Function, zero order, modified first kind.

Therefore:

$$P(R) = \frac{1}{\sigma_x\ \sigma_y} \int_{r=0}^{R} re^{-\frac{r^2}{4\sigma_y^2}\left[1 + \frac{\sigma_y^2}{\sigma_x^2}\right]} I_o\left[\frac{r^2}{4\sigma_y^2}\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right] dr \qquad (3)$$

2. Special Case of Two-Dimensional Probability Distribution Function.

When $\sigma_x = \sigma_y = \sigma_r$ (ref. nos. 18, 24), from equation (3):

$$P(R) = \frac{1}{\sigma_r^2}\int_0^R re^{-\frac{r^2}{2\sigma_r^2}} I_o\left[\frac{r^2}{4\sigma_r^2}\left(\frac{\sigma_r^2}{\sigma_r^2} - 1\right)\right] dr$$

$$P(R) = \frac{1}{\sigma_r^2}\int_0^R re^{-\frac{r^2}{2\sigma_r^2}} I_o(0)\ dr$$

$$I_o(0) = 1$$

$$P(R) = \int_0^R \frac{r}{\sigma_r^2} e^{-\frac{r^2}{2\sigma_r^2}} dr$$

Since:

$$\frac{d}{dr} e^{-\frac{r^2}{2\sigma_r^2}} = -\frac{r}{\sigma_r^2} e^{-\frac{r^2}{2\sigma_r^2}}$$

Then:

$$\int \frac{r}{\sigma_r^2} e^{-\frac{r^2}{2\sigma_r^2}} dr = -e^{-\frac{r^2}{2\sigma_r^2}}$$

$$P(R) = -e^{-\frac{r^2}{2\sigma_r^2}}\Big]_0^R = 1 - e^{-\frac{R^2}{2\sigma_r^2}}$$

$$\therefore P(R) = 1 - e^{-\frac{R^2}{2\sigma_r^2}} \qquad (4)$$

3. Modified Form of the Two-Dimensional Probability Distribution Function. (ref. no. 24) To solve equation (3) by the use of tables, the equation must be modified. From S.O. Rice's "Properties of Sine

Wave Plus Noise" Bell System Technical Journal Vol. 27 No. 1, January, 1948, pp 109-157:

$$I_e\ (kx) = \int_0^x e^{-v}\ I_o\ (vk)\ dv \tag{5}$$

Modifying equation (3):

$$\frac{1}{\sigma_x\ \sigma_y} \int_{r=0}^{R} re^{-\frac{r^2}{4\sigma_y^2}\left[1 + \frac{\sigma_y^2}{\sigma_x^2}\right]} I_o\left[\frac{r^2}{4\sigma_y^2}\ \left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right]\ dr$$

Step A

Letting:

$$v = \frac{r^2}{4\sigma_y^2}\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)$$

$$dv = \frac{2r}{4\sigma_y^2}\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)\ dr$$

$$4\sigma_y^2\ dv = 2r\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)dr$$

$$rdr = \frac{2\sigma_y^2}{\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)}\ dv$$

Step B

To get the quantity $\left[\frac{r^2}{4\sigma_y^2}\ \left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\right]$ in the form of (vk):

$$v = \frac{r^2}{4\sigma_y^2}\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)$$

$$\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right) = \left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right) k$$

$$\therefore k = \frac{\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)}{\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)}$$

Let $a = \frac{\sigma_x}{\sigma_y}$ where σ_x is the smaller of the two:

$$k = \frac{\left(\frac{\sigma_y^2}{\sigma_x^2} - \frac{\sigma_x^2}{\sigma_x^2}\right)}{\left(\frac{\sigma_x^2}{\sigma_x^2} + \frac{\sigma_y^2}{\sigma_x^2}\right)} = \frac{\left(\frac{1}{a^2} - 1\right)}{\left(1 + \frac{1}{a^2}\right)} = \frac{\left(\frac{1 - a^2}{a^2}\right)}{\left(\frac{1 + a^2}{a^2}\right)} = \frac{1 - a^2}{1 + a^2} \qquad (6)$$

Step C

Getting σ_x and σ_y in terms of a:

$$\frac{1}{\sigma_x \ \sigma_y} \cdot \frac{2\sigma_y^2}{\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)} = 2\left[\frac{1}{\sigma_x \ \sigma_y} \cdot \frac{\sigma_y^2}{\frac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}}\right] = 2\left[\frac{1}{\sigma_x \ \sigma_y} \cdot \frac{\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}\right]$$

$$2\left[\frac{\dfrac{1}{\sigma_y^2}}{\dfrac{1}{\sigma_y^2}} \cdot \frac{\dfrac{\sigma_x\ \sigma_y}{1}}{\dfrac{\sigma_x^2 + \sigma_y^2}{1}}\right] = 2\left[\frac{\dfrac{\sigma_x\ \sigma_y}{\sigma_y^2}}{\dfrac{\sigma_x^2 + \sigma_y^2}{\sigma_y^2}}\right] = 2\left[\frac{\dfrac{\sigma_x}{\sigma_y}}{\dfrac{\sigma_x^2}{\sigma_y^2} + 1}\right] = \frac{2a}{a^2 + 1} \tag{7}$$

Step D

$$1 + \frac{\sigma_y^2}{\sigma_x^2} = \frac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2} = \left[\frac{\dfrac{1}{\sigma_y^2}}{\dfrac{1}{\sigma_y^2}} \cdot \frac{\dfrac{\sigma_x^2 + \sigma_y^2}{1}}{\dfrac{\sigma_x^2}{1}}\right] = \left[\frac{\dfrac{\sigma_x^2 + \sigma_y^2}{\sigma_y^2}}{\dfrac{\sigma_x^2}{\sigma_y^2}}\right] = \frac{a^2 + 1}{a^2} \tag{8}$$

Combining Steps A, B, C, D and equation (3):

$$P(R) = \frac{2a}{1+a^2} \int_0^{\frac{R^2}{4\sigma_y^2}\left[\frac{1+a^2}{a^2}\right]} e^{-\frac{r^2}{4\sigma_y^2}\left[\frac{1+a^2}{a^2}\right]} I_o\left[\frac{r^2}{4\sigma_y^2}\left(\frac{1+a^2}{a^2}\right)\left(\frac{1-a^2}{1+a^2}\right)\right] dv. \tag{9}$$

Rewriting equation (9):

$$P(R) = \frac{2a}{1 + a^2} \int_0^x e^{-v}\ I_o\ (vk)\ dv \tag{10}$$

where:

$$x = \frac{R^2}{4\sigma_y^2}\left[\frac{1 + a^2}{a^2}\right] ; \qquad v = \frac{r^2}{4\sigma_y^2}\left[\frac{1 + a^2}{a^2}\right] ; \qquad k = \left(\frac{1 - a^2}{1 + a^2}\right)$$

4. Solution of Modified Function. (ref. nos. 12, 23) To compute the CPE (CPE ≡ R when P(R) = 0.5) for values of $\sigma_y = \sqrt{.6}$ and $\sigma_x = \sqrt{.4}$, two methods are available:

Method 1:

To determine the value for x by Rice's table of I_e (vk) dv, enter the table with values of k and the required probability.

P(R) = 50% probability; $a = \frac{\sigma_x}{\sigma_y} = 0.8165$; $a^2 = 0.6667$; $k = \frac{1-a^2}{1+a^2} = 0.2$

$$P(R) = \frac{2a}{1 + a^2} \int_0^x e^{-v} \quad I_o \, (vk) \, dv$$

$$\frac{.50 \, (1 + a^2)}{2a} = \int_0^x e^{-v} \quad I_o \, (vk) \, dv$$

$$0.5103 = \int_0^x e^{-v} \quad I_o \, (vk) \, dv$$

Enter the tables with k = 0.2 **and interpolate** for 0.5103 to get the value of x.

$$\begin{aligned} .6 &= 4517 \\ & \quad\; 5103 \\ .8 &= 5516 \end{aligned}$$

$$\therefore x = 0.71732$$

$$x = \frac{R^2}{4\sigma_y^2} \left[\frac{1 + a^2}{a^2}\right] = 0.71732$$

$$\frac{R}{\sigma_y} = 1.0713$$

The radius of the 50% probability circle (CPE) resulting from σ_x, σ_y is $R = 1.0713\ \sigma_y$.

Method 2:

Using tables computed by Arthur Grad and Herbert Solomon:

From equation (2):

$$P(R) = P\left(\sqrt{x^2 + y^2} \leq R\right) = P\left(x^2 + y^2 \leq R^2\right)$$

Since x and y have unit standard errors, they can be written as:

$x = \sigma_x\ x$ and $y = \sigma_y\ y$.

Therefore:

$$P(R) = P\left(\sigma_y^2\ y^2 + \sigma_x^2\ x^2 \leq R^2\right)$$

$$= P\left(y^2 + \frac{\sigma_x^2}{\sigma_y^2}\ x^2 \leq \frac{R^2}{\sigma_y^2}\right) \qquad (11)$$

From Grad and Solomon Tables:

$$P\left(a_1\ y_1^2 + a_2\ y_2^2 \leq t\right) \qquad a_1 + a_2 = 1$$

$$P\left(y_2^2 + \frac{a_1}{a_2}\ y_1^2 \leq \frac{t}{a_2}\right) \qquad (12)$$

Correlation between equations (11) and (12) will permit use of the tabled values.

$$\frac{\sigma_x}{\sigma_y} = \sqrt{\frac{a_1}{a_2}}\ ; \qquad \frac{R}{\sigma_y} = \sqrt{\frac{t}{a_2}}$$

Enter the tables with values of a_1, a_2 and the required probability. Then interpolate for values of $\frac{R}{\sigma_y} = \sqrt{\frac{t}{a_2}}$.

$$\text{Since } \frac{\sigma_x}{\sigma_y} = \sqrt{\frac{.4}{.6}}, \text{ then } a_1 = .4;\ a_2 = .6$$

t	=	.6	=	4559
				5000
		.7		5080

$$\frac{t}{a_2} = \frac{0.68464}{.6} = \frac{R^2}{\sigma_y^{\,2}}$$

$$\frac{R}{\sigma_y} = \sqrt{\frac{0.68464}{.6}} = 1.068$$

$$R = 1.068\ \sigma_y$$

Appendix E

DERIVATION OF THE SPHERICAL PROBABILITY DISTRIBUTION FUNCTION

The combined probability density distribution function of the independent errors x, y and z are:

$$p(x,y,z) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}} \cdot \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_y^2}} \cdot \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_z^2}} \qquad (1)$$

In the spherical case where $\sigma_x = \sigma_y = \sigma_z \equiv \sigma_s$:

$$p(x,y,z)dx\ dy\ dz = \frac{1}{\sigma_s^3 (2\pi)^{\frac{3}{2}}} e^{-\frac{x^2+y^2+z^2}{2\sigma_s^2}} dx\ dy\ dz \qquad (2)$$

Converting to 3-dimensional coordinates:

$$x^2 = s^2 \cos^2 \psi \cos^2 \lambda$$

$$y^2 = s^2 \cos^2 \psi \sin^2 \lambda$$

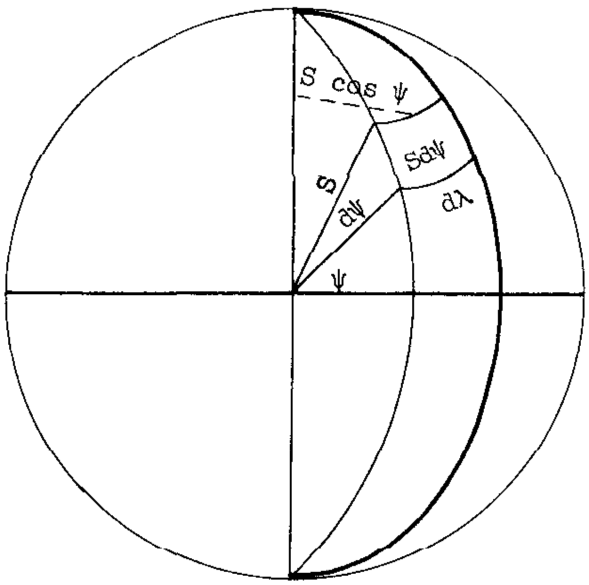
$$z^2 = s^2 \sin^2 \psi$$

$$x^2 + y^2 + z^2 = s^2 \cos^2 \psi \cos^2 \lambda + s^2 \cos^2 \psi \sin^2 \lambda + s^2 \sin^2 \psi$$

$$= s^2 \cos^2 \psi (\cos^2 \lambda + \sin^2 \lambda) + s^2 \sin^2 \psi$$

$$= s^2$$

Let S = radius of sphere, replacing radial error s.



Then: $dS \; Sd\psi \; S\cos\psi \; d\lambda \; = \; S^2 \cos\psi \; d\psi \; d\lambda \; dS$

$$P(S) = \int_{S=0}^{S} \int_{\lambda=0}^{2\pi} \int_{\psi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_s^{3}} e^{-\frac{S^2}{2\sigma_s^2}} S^2 \cos\psi \, d\psi \, d\lambda \, dS \qquad (3)$$

$$P(S) = \frac{1}{(2\pi)^{\frac{3}{2}}} (2) \, (2\pi) \int_{S=0}^{S} \frac{S^2}{\sigma_s^3} e^{-\frac{S^2}{2\sigma_s^2}} dS$$

$$\therefore P(S) = \sqrt{\frac{2}{\pi}} \int_0^S \frac{S^2}{\sigma_s^3} e^{-\frac{S^2}{2\sigma_s^2}} dS \qquad (4)$$

Integrating by parts:

Let $u = \frac{S}{\sigma_s}$, $dv = \frac{S}{\sigma_s^2} e^{-\frac{S^2}{2\sigma_s^2}} dS$

$du = \frac{dS}{\sigma_s}$, $v = -e^{-\frac{S^2}{2\sigma_s^2}}$

$$P(S) = \sqrt{\frac{2}{\pi}} \left[\left(\frac{S}{\sigma_s}\right) \left(- e^{-\frac{S^2}{2\sigma_s^2}} \right) + \int_0^S \frac{e}{\sigma_s}^{-\frac{S^2}{2\sigma_s^2}} dS \right] \tag{5}$$

In order to use approximation formula (Mathematical Tables and Other Aids to Computations, Vol. XI, No. 60, October 1957, pp 265, "A Formula for the Approximation of Definite Integrals of the Normal Distribution Function"), P(S) must be transformed to the integral of $e^{-t^2/2}$ dt.

Letting $C = \frac{S}{\sigma_s}$, $dS = \sigma_s \, dC$, where σ_s = constant:

$$P(S) = \sqrt{\frac{2}{\pi}} \left[- Ce^{-\frac{C^2}{2}} + \int_{C=0}^{C=\frac{S}{\sigma_s}} e^{-\frac{C^2}{2}} dC \right] \tag{6}$$

From above reference when $x \geq 0$:

$$\int_x^{\infty} e^{-\frac{t^2}{2}} dt \sim \frac{e^{-\frac{x^2}{2}}}{x + 0.8 \, e^{-.4x}} \tag{7}$$

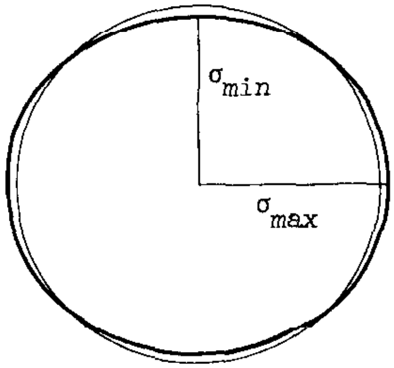
$$\int_0^{\infty} e^{-\frac{C^2}{2}} \, dC = 1.253$$

$$\int_0^{x} = \int_0^{\infty} - \int_x^{\infty}$$

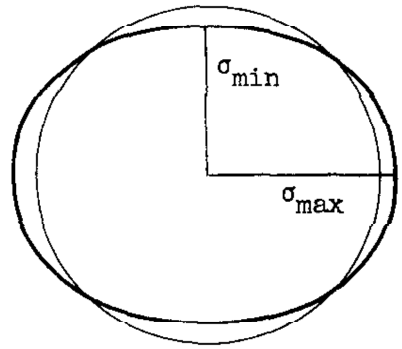
$$\therefore P(S) = \sqrt{\frac{2}{\pi}} \left[- C e^{-\frac{C^2}{2}} + 1.253 - \frac{e^{-\frac{C^2}{2}}}{C + 0.8\, e^{-.4C}} \right] \qquad (8)$$

Appendix F

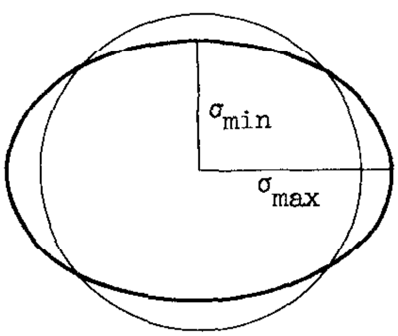
SUBSTITUTION OF THE CIRCULAR FORM FOR ELLIPTICAL ERROR DISTRIBUTIONS



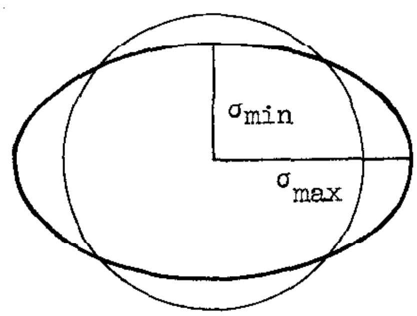
$$\frac{\sigma_{min}}{\sigma_{max}} = 0.9$$



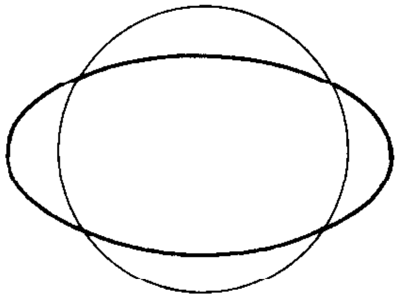
$$\frac{\sigma_{min}}{\sigma_{max}} = 0.8$$



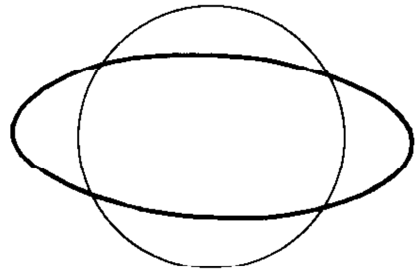
$$\frac{\sigma_{min}}{\sigma_{max}} = 0.7$$



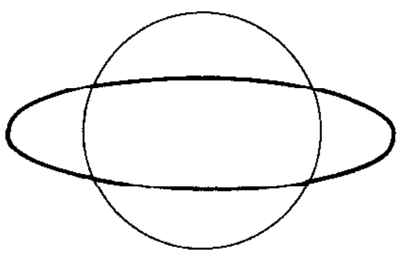
$$\frac{\sigma_{min}}{\sigma_{max}} = 0.6$$



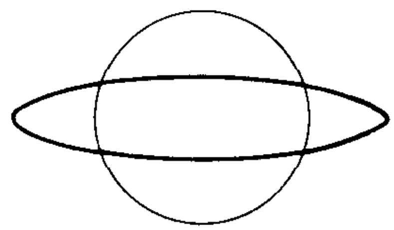
$$\frac{\sigma_{min}}{\sigma_{max}} = 0.5$$



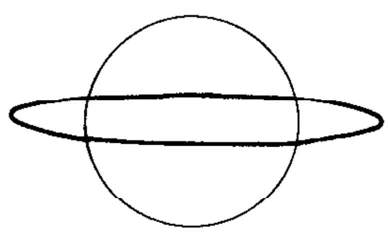
$$\frac{\sigma_{min}}{\sigma_{max}} = 0.4$$



$$\frac{\sigma_{min}}{\sigma_{max}} = 0.3$$



$$\frac{\sigma_{min}}{\sigma_{max}} = 0.2$$



$$\frac{\sigma_{min}}{\sigma_{max}} = 0.1$$

REFERENCES

1. Aeronautical Chart and Information Center Technical Report No. 95, "Positional Error: Origin, Nature, and Influence on Weapon Delivery Accuracy", October 1960.

2. Aitken, A.C., Statistical Mathematics, 5th Edition. New York: Interscience Publication, Inc., 1945.

3. Arley, Niels and K. Rander Buch, Introduction to the Theory of Probability and Statistics. New York: John Wiley and Sons, Inc., 1950.

4. Bartlett, Dana P., General Principles of the Method of Least Squares with Application, 3rd Edition. Boston: The Author, 1915.

5. Beers, Yardley, Introduction to the Theory of Errors, 2nd Edition. Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1958.

6. Bomford, Brigadier G., Geodesy. Oxford: Clarendon Press, 1952.

7. Bouchard, Harry, revised by Francies H. Moffitt, Surveying, 4th Edition. Scranton, Pa.: International Text Book Co., 1889, revised in 1959.

8. Cramer, H., Mathematical Methods of Statistics, Stockholm and Princeton: Princeton University Press, 1946.

9. Crum, W.L., Rudimentary Mathematics for Economist and Statisticians, London: McGraw-Hill Book Co., 1946.

10. Freund, John E., Modern Elementary Statistics, 2nd Printing. New York: Prentice-Hall, Inc., 1953.

11. Gaida, Joseph, "Error Theory for Evaluation of ACIC Products", unpublished document.

12. Grad, Arthur and Herbert Solomon, "Distribution of Quadratic Forms and Some Applications", The Annuals of Mathematical Statistics, Vol. 26, pp 464-477. 1955.

13. Hart, Roger G., "A Formula for the Approximation of Definite Integrals of the Normal Distribution Function", Mathematical Tables and Other Aids to Computation, Vol. XI, No. 60, p. 265. October 1957.

14. Hosmer, George L., Geodesy, 2nd Edition. New York: John Wiley and Sons, Inc., 1946.

15. Jordan, W. and O. Eggert, Handbuch der Vermessungkunde, Vol. 1, Chap. 1 and 5. Stuttgart, 1948.

16. Kissam, Phillip, Surveying for Civil Engineers. New York: McGraw-Hill, 1956.

17. Levey, Hyman and E.E. Preidel, Elementary Statistics. New York: Ronald Press Co., 1945.

18. Levine, Daniel, Radargrammetry. New York: McGraw-Hill, 1960.

19. Lyon, Duane, Basic Metrical Photogrammetry, Revised. St. Louis: John S. Swift Co., 1960.

20. Merrill, Grayson, editor, Principles of Guided Missile Design, "Operations Research, Armament, and Launching," pp. 97-111; "Guidance", pp. 296-301. Princeton, New Jersey, Toronto, New York, London: D. Van Nostrand Co., 1956.

21. McGraw-Hill, Encylopedia of Science and Technology, 1960.

22. Rainsford, Hume F., Survey Adjustments and Least Squares. New York: Fredrick Ungar Publishing Co., 1958.

23. Rice, S.O., "Properties of Sine Wave Plus Noise", Bell System Technical Journal, Vol. 27, No. 1, pp. 109-157. January 1948.

24. Space Technology Laboratories, Inc., Ballistic Missiles, Vol. IV, Revision 1. Los Angeles 45, California, 1960.

25. Stearn, Joseph L., "Some Fundamentals of Error Theory", The Journal U.S. Coast and Geodetic Survey, No. 5, pp. 79-84. June 1953.

26. U.S. Army Field Manual 101-131, "Probability Considerations", Chap. 7.

27. U.S. Army Ordnance Missile Command, "Geodetic Requirements for Inertially Guided Army Ballistic Missile Systems (U)", October 1960 (Classified).

28. Walker, Helen M., Elementary Statistical Methods. Henry Holt.

29. Worthington, Archie G., Treatment of Experimental Data. New York: John Wiley and Sons, Inc., 1959.